1 \( \ell \)-Way Cut / Max \( \ell \)-Cut Problem

**Input:** \( G=(V,E) \) (assume unweighted for simplicity), \( n=|V| \).

**Goal:** Find partition \( S = S_1, \ldots, S_\ell \) of \( V \) maximizing \( |E(S)| \) where

\[
E(S) = \{ \{ u, v \} : u \in S_i, v \in S_j \text{ for some } i \neq j \}.
\]

This is another problem for which we do not know an algorithm that outputs an optimal solution, but as we will show, our algorithm can output a solution that is “close” to optimal. Before we can express our algorithm, we need to set up some notation and state an important lemma.

So let \( G = (V,E) \) and \( A, B \subseteq V \), then let \( e(A,B) = |E(A,B)| \) where \( E(A,B) \) is the set of edges between \( A \) and \( B \). Now let \( d(A,B) = \frac{e(A,B)}{|A||B|} \). Now we can state the following definition.

**Definition 1.** Suppose \( A \cap B = \emptyset \). Then we say that \( (A,B) \) is \( \varepsilon \)-regular if for all \( X \subseteq A \) with \( |X| \geq \varepsilon |A| \) and for all \( Y \subseteq B \) with \( |Y| \geq \varepsilon |B| \), we have \( |d(X,Y) - d(A,B)| < \varepsilon \).

With this notation and definition at our disposal we can now state Szemeredi’s Regularity Lemma.

**Lemma 1** (Szemeredi’s Regularity Lemma). For all \( \varepsilon > 0 \), for all \( m \in \mathbb{Z}^+ \), there exists \( P(\varepsilon,m), Q(\varepsilon,m) \in \mathbb{Z} \) such that for all graphs \( G = (V,E) \) with \( n \geq P(\varepsilon,m) \) there exists partition \( V_1, \ldots, V_k \) of \( V \) such that

i. \( m \leq k \leq Q(\varepsilon,m) \);

ii. \( \left\lceil \frac{n}{k} \right\rceil - 1 \leq |V_i| \leq \left\lceil \frac{n}{k} \right\rceil \);

iii. All but \( \varepsilon k^2 \) of the pairs \( (V_i,V_j) \) are \( \varepsilon \)-regular.

**Remark.** Partitions that satisfy iii. in Szemeredi’s Regularity Lemma are called \( \varepsilon \)-regular partitions.
Now let us develop some more notation. Let $V_1, \ldots, V_k$ be a partition of $V$, $K = \{1, \ldots, k\}$, and $d_{i,j} = d(V_i, V_j)$. For $X \subseteq V$, $I \subseteq K$, let $X_I = \bigcup_{i \in I} X_i$ where $X_i = X \cap V_i$. Let $S, T \subseteq V$ such that $S \cap T = \emptyset$. Let
\[
\Delta(S, T) = e(S, T) - \sum_{i \in K} \sum_{j \in K} d_{i,j} \cdot |S_i| \cdot |T_j|.
\]

**Remark.** If $(V_i, V_j)$ is $\varepsilon$-regular then $e(S_i, T_j) \approx d_{i,j} \cdot |S_i| \cdot |T_j|$. In other words, $\Delta(S, T)$ measures the “deviation from regularity”.

**Definition 2.** We say that $V_1, \ldots, V_k$ is $\varepsilon$-sufficient if $|\Delta(S, T)| \leq \varepsilon n^2$ for all $S, t \subset V$ with $S \cap T = \emptyset$.

The following lemma will tell us that as long as $k$ is large enough the partition given by Szemeredi’s Regularity Lemma is also $4\varepsilon$-sufficient.

**Lemma 2.** An $\varepsilon$-regular partition with $k \geq \frac{1}{\varepsilon}$ is $4\varepsilon$-sufficient.

**Proof.** Suppose $V_1, \ldots, V_k$ is $\varepsilon$-regular partition and $v = \lceil \frac{n}{k} \rceil$ where $n, k$ are as defined in Szemeredi’s Regularity Lemma. Let $S, T \subseteq V$ such that $S \cap T = \emptyset$ and let
\[
L_2 = \{(i, j) \in K \times K : |S_i| \leq \varepsilon v \text{ or } |T_j| \leq \varepsilon v\},
\]
\[
L = \{(i, j) \in K \times K : i \neq j \text{ and } (V_i, V_j) \text{ is } \varepsilon \text{-regular}\},
\]
\[
L_1 = L \setminus L_2, L_3 = (K \times K) \setminus (L_1 \cup L_2), \text{ and } L_4 = \{(i, i) : i \in K\}
\]
Then $\Delta(S, T) = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$ where $\Delta_i = \sum_{(i,j) \in L_i} (e(S_i, T_i) - \sum_{j \in K} d_{i,j} \cdot |S_i| \cdot |T_j|)$. So then we have that for all $i \in \{1, 2, 3, 4\}$, $\Delta_i \leq \varepsilon r^2 k^2$ and so $\Delta(S, T) \leq 4\varepsilon n^2$. Thus, the partition is $4\varepsilon$-regular.

An important side-note that we’ve been omitting is if these $\varepsilon$-regular partitions can be computed in a reasonable amount of time. Szemeredi’s Regularity Lemma tells us that they exist but not necessarily that we can construct them efficiently. Luckily, our next theorem does.

**Theorem 1** (Alon, Duke, Lehmann, Rodd, Yuster). An $\varepsilon$-regular partition can be efficiently computed.

The following theorem solves the problem with a close to optimal partition.

**Theorem 2.** There is a randomized polynomial time algorithm which given an $n$-vertex graph $G$, with probability at least $3/4$, computes a partition $S_\varepsilon$ such that $|E(S_\varepsilon)| \geq |E(S^*)| - \varepsilon n^2$ where $S^*$ is an optimal partition.