1 \(\ell\)-Way Cut Continued

Recall from previous lectures the following problem statement

\[ \text{\(\ell\)-Way Cut:}
\begin{align*}
\text{Input:} & \quad G = (V, E), \quad n = |V| \\
\text{Goal:} & \quad \text{Find a partition } S_1, \ldots, S_\ell \text{ of } V \text{ maximizing } |E(S)|, \text{ where}
\end{align*}
\]

\[ E(S) = \{\{u, v\} \in E : u \in S_i \text{ and } v \in S_j \text{ for } i \neq j\} \]

and the following definition

**Definition 1.1.** We say that a partition \(V_1, \ldots, V_k\) of \(V\) is \("\epsilon\)-sufficient" if

\[ |\Delta(S, T)| \leq \epsilon \cdot n^2, \quad \forall S, T \subseteq V, S \cap T = \emptyset, \]

where

- \(\Delta(S, T) = e(S, T) - \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j}|S_i||T_i|\)
- \(e(S, T) = |E(S, T)|\)
- \(d_{i,j} = d(V_i, V_j) = e(V_i, V_j)/|V_i||V_j|\).

Consider the example of Figure 1. Here we have the following:

- \(e(S, T) = 3\)
- \(d_{1,1} = d_{2,2} = 0\)
- \(d_{1,2} = d_{2,1} = 4^2/4^2 = 1\)

Thus, in this case we have that

\[ \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j}|S_i||T_i| = d_{1,2}|S_1||T_2| + d_{2,1}|S_2||T_s| = 2 \cdot 1 + 1 \cdot 1 = 3 \]

and therefore Figure 1 is 0-sufficient.
Theorem 1.2. There is a randomized polynomial time algorithm which given an \( n \)-vertex graph \( G \), with probability at least \( \frac{3}{4} \), computes a partition \( S_\epsilon \) such that
\[
|E(S_\epsilon)| \geq |E(S^*)| - \epsilon n^2,
\]
where \( S^* \) is an optimal partition.

Proof. Compute an \( \epsilon \)-sufficient partition \( V_1, \ldots, V_k \) of \( G \) using [Alon et al.]. Let \( S = S_1, \ldots, S_\ell \) be an \( \ell \)-way cut of \( G \). Let \( S_{i,r} = S_r \cap V_i \), and \( T_{i,r} = V_i \setminus S_{i,r} \). We have that
\[
2 |E(S)| = \sum_{r=1}^{\ell} e(S_r, V \setminus S_r).
\]
By the definition of \( \epsilon \)-sufficient, we have that
\[
2 |E(S)| = \left( \sum_{r=1}^{\ell} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} |S_{i,r}| |T_{j,r}| \right) + \Theta
\]
where \( \Theta \leq \ell \epsilon n^2 \). Let \( v_j = |V_j|, \rho = \left\lfloor \frac{\epsilon n}{k} \right\rfloor, \bar{v}_j = \left\lfloor \frac{v_j}{\rho} \right\rfloor, n_{i,r} = |S_{i,r}|, \bar{n}_{i,r} = \left\lfloor \frac{n_{i,r}}{\rho} \right\rfloor \). We have
\[
|n_{i,r}(v_j - n_{j,r}) - \rho^2 \bar{n}_{i,r}(\bar{v}_j - \bar{n}_{j,r})| \leq \rho(v_i + v_j).
\]
Thus
\[
\left| \sum_{r=1}^{\ell} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} n_{i,r}(v_j - n_{j,r}) - \rho^2 \sum_{r=1}^{\ell} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} \bar{n}_{i,r}(\bar{v}_j + \bar{n}_{j,r}) \right| \leq 2\epsilon \ell n^2.
\]
Therefore,
\[
\left| 2|E(S)| - \rho^2 \sum_{r=1}^{\ell} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} \bar{n}_{i,r}(\bar{v}_j + \bar{n}_{j,r}) \right| \leq 3\epsilon \ell n^2.
\]
Each $\bar{n}_{i,r}$ has at most $\frac{1}{\epsilon}$ different possible values. There are $k\ell$ variables $n_{i,r}$. Therefore, there are $(\frac{1}{\epsilon})^{k\ell}$ possibilities. It is sufficient to iterate over all these possibilities, and select the instance maximizing

$$\sum_{r=1}^{\ell} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} \bar{n}_{i,r}(\bar{v}_j + \bar{n}_{j,r}).$$

Recall that we have the following lemma from a previous lecture:

**Lemma 1.3.** If $|E| \geq \gamma n^2$ then $|E(S^*)| \geq (1 - \frac{1}{\ell})|E| \geq \gamma (1 - \frac{1}{\ell})n^2$.

Using this lemma and the theorem above, we have the following corollary:

**Corollary 1.4.** For all $\epsilon > 0$ there exists a polynomial time algorithm that computes a $(1 + \epsilon)$-approximation $\ell$-way cut on dense graphs.

Thus, we have a **PTAS (Polynomial Time Approximation Scheme)** for $\ell$-Way Cut on dense graphs.