On graph crossing number and edge planarization

Julia Chuzhoy (TTIC)
Yury Makarychev (TTIC)
Anastasios Sidiropoulos (TTIC)
Computational topology

• Computation on topological objects
  – Algorithms on topologically simple input
  – Recognition of topological invariants

• Dimensionality
  – dim = 1: Trivial
  – dim = 2: Most of known computational topology. (planar graphs, graphs on surfaces, 2-manifolds, etc)
  – dim = 3: Most problems open. (knots, recognition of 3-manifolds, etc)
  – Higher dimensions: most problems intractable / undecidable
2-dimensional topological invariants

- Planarity
- Genus
- Crossing number
- Min edge/vertex planarization
Recognition of topological invariants

• Exact algorithms
  – Planarity testing [Hopcroft,Tarjan’74]
  – Genus [Mohar’99], [Kawarabayashi,Mohar,Reed’08]
  – Graph minor theorem [Robertson,Seymour’99]
  – Crossing Number [Kawarabayashi,Reed’07], [Grohe’04]

• Approximation algorithms
  – Partial results on Crossing Number
Crossing number

\[ cr(K_5) = 1 \]

\[ cr(K_{13}) = ? \]
Computing the crossing number

- **> 600** papers on Crossing Number [Vrt’o]
- No PTAS for $cr(G)$. [Garey,Johnson’83], [Ambuhl,Mastrolilli,Svensson’07]
- Linear-time algorithm for fixed $cr(G)$. [Kawarabayashi,Reed’07], [Grohe’04]
- $O(n \cdot \log^2 n)$-approximation [Even,Guha,Schieber’02], [Leighton,Rao’92], [Bhatt,Leighton’84], [Arora,Rao,Vazirani’09]
- Better approximations for special classes
  - Planar + one edge: NP-hard [Cabello,Mohar’10], $O(1)$-approx. [Hlineny,Salazar’07]
  - 1-apex graphs: $O(1)$-approx. [Chimani,Hlineny,Mutzel’09]
  - Toroidal graphs: $O(1)$-approx. [Hlineny,Salazar’07]
  - Projective graphs [Gitler,Hlineny,Leanos,Salazar’07]
  - Some small genus graphs [Hlineny,Chimani’10]
Our results

- **Theorem** [Chuzhoy, Makarychev, S ’11]
  Given a graph $G = \text{planar} + k$ edges, we can find a drawing with $O(k \cdot (k + cr(G)))$ crossings.

- **Corollary** [Chuzhoy, Makarychev, S ’11]
  $O(n \cdot \log^{3/2} n)$-approximation for $cr(G)$.

- **Corollary** [Chuzhoy, Makarychev, S ’11]
  $2^{O(g)} \cdot n^{1/2}$-approximation for $cr(G)$ on genus-$g$ graphs.

- **Corollary** [Chuzhoy, Makarychev, S ‘11]
  $O(1)$-approximation for $O(1)$-apex graphs.
Further implications

• Every graph can be made planar by removing $cr(G)$ edges.

• **Corollary [Chuzhoy, Makarychev, S ’11]**
  Any approximation for Min-Edge-Planarization implies an approximation for $cr(G)$.

• **Theorem [Chuzhoy]**
  $OPT^{O(1)}$-approximation Min-Edge-Planarization.

• **Corollary**
  $O(n^{1-\alpha})$-approximation for $cr(G)$. 
$O(n \cdot \log^2 n)$-approximation for $cr(G)$

- **Main idea:** [Bhatt, Leighton’84], [Leighton, Rao’92], [Even, Guha, Schieber’02]
  - Planar graphs have small separators. [Lipton, Tarjan’79]
  - Small crossing number implies small separators.
  - Divide & conquer.
$O(n \cdot \log^2 n)$-approximation for $cr(G)$
Our approach

Key idea:

Any drawing of a 3-connected graph with a few crossings, is "close" to a planar drawing.
Rotation systems

• For every vertex, specify the cyclic ordering of its adjacent edges

1
2
3

≠

1
2
3
Local vs global

• 3-connected planar graphs, have a unique drawing. [Whitney ‘32]

• All planar drawings of a 3-connected planar graph have the same rotation system, up to orientation.
Example
Approximate rotation systems

• What about non-planar drawings?
  – Do all drawings with a few crossings have “similar” rotation systems?
  – For drawings $\varphi, \psi$ let $\text{IRG}(\varphi, \psi)$ be the set of vertices with different orderings in the corresponding rotation systems.

Lemma [Chuzhoy, Makarychev, S’11]
Let $G$ be a 3-connected planar graph.
Let $\varphi$ be the unique planar drawing of $G$.
Let $\psi$ be a drawing of $G$ with $s$ crossings.
Then, $\text{IRG}(\varphi, \psi) = O(s)$. 
Approximate rotation systems (example)
The main argument

\[ G = \text{planar } H + k \text{ edges.} \]

Assume \( H \) is 3-connected.
Routing along paths

Draw every extra edge “close” to a shortest path in the dual graph.
The main argument (cont.)

Route extra edges along paths of $H$. $O(k^2)$ crossings between the routing paths. Suffices to bound crossings between routing paths and $H$. 
The main argument (cont.)

Lemma:
Every routing path has at most $O(cr(G))$ more crossings in $\psi$ compared to $OPT$. 
The general case

- $H$ might not be 3-connected
- We need to find a planar drawing of $H$, with a rotation system similar to the optimal.
- Block decomposition, similar to SPQR trees.
- Many technical details...
Further directions

• $G = \textit{planar} + k \textit{ edges}$, is there an $O(k)$-approx. for $cr(G)$?
• $O(1)$-approx. for $cr(G)$?
• Other topological parameters?
• Approximate versions of the Graph Minor Theorem?
• We know a few separations.
  – Clique minors: $\Omega(n^{1/2})$-hard [Alon,Lingas,Wahlen’07], fixed-parameter tracktable [Robertson,Seymour’86]