On-line embeddings

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Abstract

We initiate the study of on-line metric embeddings. In such an embedding we are given a sequence of $n$ points $X = x_1, \ldots, x_n$ one by one, from a metric space $M = (X, D)$. Our goal is to compute a low-distortion embedding of $M$ into some host space, which has to be constructed in an on-line fashion, so that the image of each $x_i$ depends only on $x_1, \ldots, x_i$. We prove several results translating existing embeddings to the on-line setting, for the case of embedding into $\ell_p$ spaces, and into distributions over ultrametrics.

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1 Introduction

A low-distortion (or bi-Lipschitz) embedding between two metric spaces \( M = (X, D) \) and \( M' = (X', D') \) is a mapping \( f \) such that for any pair of points \( p, q \in X \) we have \( D(p, q) \leq D'(f(p), f(q)) \leq c \cdot D(p, q) \); the factor \( c \) is called the distortion of \( f \). In recent years, low-distortion embeddings found numerous applications in computer science [17][10]. This can be, in part, attributed to the fact that embeddings provide a general method for designing approximation algorithms for problems defined over a “hard” metric, by embedding the input into an “easy” metric and solving the problem in there.

For some problems, however, applying this paradigm encounters difficulties. Consider for example the nearest neighbor problem: given a set \( P \) of \( n \) points in some metric \( (X, D) \), the goal is to build a data structure that finds the nearest point in \( P \) to a query point \( q \in X \). A fundamental theorem of Bourgain [4] shows that it is possible to embed \( P \) and the query point \( q \) into an “easy” metric space, such as \( \ell_2 \) with distortion \( \log n \). This, however, does not translate to an efficient approximation algorithm for the problem for the simple reason that the query point \( q \) is not known at the preprocessing stage, so it cannot be embedded together with the set \( P \). More specifically, for the approach to work in this scenario we must require that we can extend the embeddings \( f: P \rightarrow \ell_2 \) to \( g: P \cup \{q\} \rightarrow \ell_2 \). We note that the aforementioned Bourgain’s theorem [4] does not have such an extendability property.

An even more straightforward setting in which the standard notion of embeddings is not quite the right notion comes up in the design of on-line algorithms. Often, the input considered is metric space; at each step the algorithm receives an input point and needs to make decisions about it instantly. In order to use the embedding method, we must require that the embedding would observe the inputs sequentially, so that a point is mapped based only on the distance information of the points observed so far. Here is a precise definition of the desired object.

Definition 1. An on-line embedding of an \( n \)-point metric space \( M = (X, D) \) where \( X = \{x_1, \ldots, x_n\} \) into some host metric space \( M' \) is a sequence of functions \( f_k \) for \( k = 1, \ldots, n \) (possibly randomized) such that

- \( f_k \) depends only on \( M_k \), the restriction of \( M \) on \( \{x_1, \ldots, x_k\} \).
- \( f_k \) extends \( f_{k-1} \): for each \( x \in \{x_1, \ldots, x_{k-1}\} \), \( f_k(x) = f_{k-1}(x) \). If the functions are randomized, the extendability property means that the random bits used for \( f_{k-1} \) are a subset of the random bits for \( f_k \), and when these bits between \( f_{k-1} \) and \( f_k \) coincide the (deterministic) image of \( x \in \{x_1, \ldots, x_{k-1}\} \) is the same for these functions.

The associated distortion of the above \( f_1, \ldots, f_n \) is the distortion of \( f_n \). If \( f_i \) can be obtained algorithmically, then we say that we have an on-line algorithm for the embedding problem. We also consider on-line embeddings into shortest-path metrics of graphs. In this case, we require that \( M_k \) is mapped into a graph \( G_k \), and that every \( G_k \) is subgraph of \( G_{k+1} \).

In this work we investigate fundamental embedding questions in the on-line context. Can we hope, for example, to embed a general metric space in Euclidean space in an on-line fashion? Not surprisingly, the use of randomization is almost always essential in the design of such embeddings. It is interesting to relate the above notion to “oblivious embeddings”. An embedding is said to be oblivious, if the image of a point does not depend on other points. In the usual (off-line) embeddings, the image of a point may depend on all other points. In this language, on-line embedding is some type of middle-ground between these two types of embeddings. In particular, oblivious embeddings are a special, very restricted case of on-line embedding. Oblivious embeddings play an important role in the design of algorithms, for example in the context of streaming algorithms [12] or in the design of near linear algorithms that rely on embeddings [1]. Indeed,
some of our results use oblivious embeddings as a building block, most notably, random projections and construction of random decompositions.

1.1 Results and motivation

Embedding into $\ell_p$ spaces, and into distributions over ultrametrics. We start our investigation by considering embeddings into $\ell_p$ spaces, and into distributions over ultrametrics. These target spaces have been studied extensively in the embedding literature.

We observe that Bartal’s embedding [2] can be easily modified to work in the on-line setting. We remark that this observation was also made independently by Englert, Räcke, and Westermann [6]. As a consequence, we obtain an on-line analog of Bourgain’s theorem [4]. More specifically, we deduce that any $n$-point metric space with spread $\Delta$ can be embedded on-line into $\ell_p$ with distortion $O((\log \Delta)^{1/p} \log n)$. Similarly, we also obtain an analog of a theorem due to Bartal [2] for embedding into ultrametrics. More precisely, we give an on-line probabilistic embedding of an input metric into a distribution over ultrametrics with distortion $O((\log n \cdot \log \Delta))$.

Doubling metrics. For the special case when the input space is doubling, we give an improved on-line embedding into ultrametrics with distortion $O(\log \Delta)$. We complement this upper bound by exhibiting a distribution $F$ over doubling metrics (in fact, subsets of $\mathbb{R}^1$) such that any on-line embedding of a metric chosen from $F$ into ultrametrics has distortion $\Omega(\min\{n, \log \Delta\})$.

Embedding into $\ell_\infty$. We also consider on-line analogs of another embedding theorem, due to Fréchet, which states that any $n$-point metric can be embedded into $\ell_\infty$ with distortion 1. We show that this theorem extends to the on-line setting with the same distortion, albeit larger dimension. By composing our on-line embedding into $\ell_2$, with a random projection, we obtain for any $\alpha > \sqrt{2}$, an on-line embedding into $\ell_\infty$ with distortion $O(\alpha \cdot \log n \sqrt{\log \Delta})$, and dimension $\Omega(\max\{\max\{\log n, \log \Delta\}, \frac{n^4}{(\alpha^2 - 2)}\})$.

On-line embedding when an (off-line) isometry or near-isometry is possible. Finally, we consider the case of embedding into constant-dimensional $\ell_p$ spaces. It is well known [18] that for any constant dimension there are spaces that require polynomial distortion (e.g. via a simple volume argument). It is therefore natural to study the embedding question for instances that do embed with small distortion. When a metric embeds isometrically into $\ell_2$ or $\ell^d_2$, it is clear that this isometry can be found on-line. We exhibit a sharp contrast with this simple fact for the case when there is only a near-isometry guaranteed. Using a topological argument, we prove that there exists a distribution $D$ over metric spaces that $(1 + \varepsilon)$-embed into $\ell^d_2$, yet any on-line algorithm with input drawn from $D$ computes an embedding with distortion $n^{\Omega(1/d)}$. In light of our positive results about embedding into $\ell_2$ and a result of Matoušek [18], this bound can be shown to be a near-optimal for on-line embeddings.

Remark 1. For simplicity of the exposition, we will assume that $n$ is given to the on-line algorithm in advance. We remark however that with the single exception of embedding into $\ell_\infty$, all of our algorithms can be modified so work without this knowledge.

1.2 Related work

The notion of low-distortion on-line embeddings is related to the well-studied notion of Lipschitz extensions. A prototypical question in the latter area is: for spaces $Y$ and $Z$, is it true that for every $X \subset Y$, and every $C$-Lipschitz mapping $f : X \to Z$ it is possible to extend $f$ to $f' : Y \to Z$ which is $C'$-Lipschitz, for $C'$

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1. The ratio between the largest and the smallest non-zero distances in the metric space.
2. I.e., a mapping which expands the distances by a factor at most $C$. 

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not much greater than $C$? For many classes of metric spaces the answer to this question is positive (e.g., see the overview in [16]).

One could ask if analogous theorems hold for low-distortion (i.e., bi-Lipschitz) mapping. If so, we could try to construct on-line embeddings by repeatedly constructing bi-Lipschitz extensions to points $p_1, p_2, \ldots$. Unfortunately, bi-Lipschitz extension theorems are more rare, since the constraints are much more stringent.

In the context of the aforementioned work, the on-line embeddings can be viewed as “weak” bi-Lipschitz extension theorems, which hold for only some mappings $f : X \to Z, X \subset Y$.

### 1.3 Notation and definitions

For a point $y \in \mathbb{R}^d$, we denote by $y_i$ the $i$-th coordinate of $y$. That is, $y = (y_1, \ldots, y_d)$. Similarly, for a function $f : A \to \mathbb{R}^d$, and for $a \in A$, we use the notation $f(a) = (f_1(a), \ldots, f_d(a))$. Also, we denote by $\ell_p$ the space of sequences with finite $p$-norm, i.e., $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} < \infty$.

Consider a finite metric space $(X, D)$ and let $n = |X|$. For any point $x \in X$ and $r \geq 0$, the ball with radius $r$ around $x$ is defined as $B_X(x, r) = \{z \in X \mid D(x, z) \leq r\}$. We omit the subscript when it is clear from the context. A metric space $(X, D)$ is called $\Lambda$-doubling if for any $x \in X, r \geq 0$ the ball $B(x, r)$ can be covered by $\Lambda$ balls of radius $r/2$. The doubling constant of $X$ is the infimum $\Lambda$ so that $X$ is $\Lambda$-doubling. The doubling dimension of $X$ is $\dim(X) = \log_2 \Lambda$. A metric space with $\dim(X) = O(1)$ is called doubling. A $\gamma$-net for a metric space $(X, D)$ is a set $N \subseteq X$ such that for any $x, y \in N$, $D_X(x, y) \geq \gamma$ and $X \subseteq \cup_{x \in N} B_X(x, \gamma)$. Let $M_1 = (X, D_1)$ and $M_2 = (X, D_2)$ be two metric spaces. We say that $M_1$ dominates $M_2$ if for every $i, j \in X$, $D_1(i, j) \geq D_2(i, j)$. Let $(X, D_1)$ and $(Y, D_2)$ be two metric spaces and an embedding $f : X \to Y$. We say that $f$ is non-expanding if $f$ doesn’t expand distances between every pair $x_1, x_2 \in X$, i.e., $D_2(f(x_1), f(x_2)) \leq D_1(x_1, x_2)$. Similarly, $f$ is non-contracting if it doesn’t contract pair-wise distances. Also we say that $f$ is $\alpha$-bi-Lipschitz if there exists $\beta > 0$ such that for every $x_1, x_2 \in X, \beta D_1(x_1, x_2) \leq D_2(f(x_1), f(x_2)) \leq \alpha \beta D_1(x_1, x_2)$.

## 2 Embedding general metrics into ultrametrics and into $\ell_p$

In this section we will describe an on-line algorithm for embedding arbitrary metrics into $\ell_p$, with distortion $O(\log n \cdot (\log \Delta)^{1/p})$, for any $p \in [1, \infty]$. We also give an on-line probabilistic embedding into a distribution
over ultrametrics with distortion $O(\log n \cdot \log \Delta)$. Both algorithms are on-line versions of the algorithm of Bartal [2], for embedding metrics into a distribution of dominating HSTs, with distortion $O(\log^2 n)$. Before we describe the algorithm we need to introduce some notation.

**Definition 2 ([2]).** An l-partition of a metric $M = (X, D)$ is a partition $Y_1, \ldots, Y_l$ of $X$, such that the diameter of each $Y_i$ is at most l.

For a distribution $\mathcal{F}$ over l-partitions of a metric $M = (X, D)$, and for $u, v \in X$, let $p_{\mathcal{F}}(u, v)$ denote the probability that in an l-partition chosen from $\mathcal{F}$, $u$ and $v$ belong to different clusters.

**Definition 3 ([2]).** An $(r, \rho, \lambda)$-probabilistic partition of a metric $M = (X, D)$ is a probability distribution $\mathcal{F}$ over r-prob-partitions of $M$, such that for each $u, v \in X$, $p_{\mathcal{F}}(u, v) \leq \lambda \frac{D(u, v)}{r}$. Moreover, $\mathcal{F}$ is forcing if for any $u, v \in X$, with $D(u, v) \leq \varepsilon \cdot r$, we have $p_{\mathcal{F}}(u, v) = 0$.

We observe that Bartal’s algorithm [2] can be interpreted as an on-line algorithm for constructing probabilistic partitions. The input to the problem is a metric $M = (X, D)$, and a parameter $r$. In the first step of Bartal’s algorithm, every edge of length less than $\frac{1}{n}$ is contracted. This step cannot be directly performed in an on-line setting, and this is the reason that the parameters of our probabilistic partition will depend on $\Delta$. More precisely, our partition will be $1/\Delta$-forcing, while the one obtained by Bartal’s off-line algorithm is $1/n$-forcing.

The algorithm proceeds as follows. We begin with an empty partition $P$. At every step $j$, each $Y_i \in P$ will correspond to a ball of some fixed radius $r_i$ around a point $y_i \in X_j$. Once we have picked $y_i$, and $r_i$, they will remain fixed until the end of the algorithm. Assume that we have partitioned all the points $x_1, \ldots, x_{i-1}$, and that we receive $x_i$. Let $P = \{Y_1, \ldots, Y_k\}$. If $x_i \notin \bigcup_{j \in [k]} B(y_j, r_j)$, then we add a new cluster $Y_{k+1}$ in $P$, with center $y_{k+1} = x_i$, and we pick the radius $r_{k+1} \in [0, r \log n)$, according to the probability distribution $p(r_{k+1}) = \left(\frac{n}{n-1}\right)^{1/r} e^{-r_{k+1}/r}$. Otherwise, let $Y_s$ be the minimum-index cluster in $P$, such that $x_i \in B(y_s, r_s)$, and add $x_i$ to $Y_s$.

By Bartal’s analysis on the above procedure, we obtain the following lemma.

**Lemma 1.** Let $M$ be a metric, and $r \in [1, \Delta]$. There exists an $1/\Delta$-forcing, $(r, O(\log n), O(1))$-probabilistic partition $\mathcal{F}$ of $M$, and a randomized on-line algorithm that against any non-adaptive adversary, given $M$ computes a partition $P$ distributed according to $\mathcal{F}$. Moreover, after each step $i$, the algorithm computes the restriction of $P$ on $X_i$.

By the above discussion it follows that for any $r > 0$ we can compute an $(r, O(\log n), O(1))$-probabilistic partition of the input space $M = (X, D)$. It is well known that this implies an embedding into $\ell_p$ for any $p \in [1, \infty]$. Since the construction is folklore (see e.g. [22, 2, 8]), we will only give a brief overview, demonstrating that the embedding can be indeed computed in an on-line fashion.

For each $i \in \{1, \ldots, \log \Delta\}$, and for each $j \in \{1, \ldots, O(\log n)\}$ we sample a probabilistic partition $P_{i,j}$ of $M$ with clusters of radius $2^i$. Each such cluster corresponds to a subset of a ball of radius $2^i$ centered at some point of $M$. For every $i$, we compute a mapping $f_{i,j} : X \to \mathbb{R}$ as follows. For each cluster $C \in P_{i,j}$ we choose $s_{i,j} \in \{-1, 1\}$ uniformly at random. Next, for each point $x \in X$ we need to compute its distance $h_{i,j}(x)$ to the “boundary” of the union of all clusters. For every $C \in P_{i,j}$ let $a(C)$, $r(C)$ be the center and radius of $C$, respectively. We can order the clusters in $(C_1, \ldots, C_k)$, so that $C_t$ is created by the on-line algorithm before $C_t$ for every $t < l$. For a point $x \in X$ we let $C(x)$ be the cluster containing $x$. Suppose $C(x) = C_t$. We set $h_{i,j}(x) = \min_{t \in \{1, \ldots, t\}} |r(C_t) - D(x, a(C_t))|$. Note that $h_{i,j}(x)$ can be computed in an on-line fashion. We set $f_{i,j}(x) = s_{i,j} \cdot h_{i,j}(x)$. The resulting embedding is
computes an embedding of a given metric into \( \ell_p \). It is now straightforward to verify that with high probability, for all \( x, y \in X \) we have
\[
D(x, y) \cdot \Omega((\log n)^{1/p} / \log n) \geq \|\varphi(x) - \varphi(y)\|_p \geq D(x, y) \cdot \Omega((\log n)^{1/p} / \log n),
\]
implying the following result.

**Theorem 1.** There exists an on-line algorithm that for any \( p \in [1, \infty] \), against a non-adaptive adversary, computes an embedding of a given metric into \( \ell_p^{O(\log n \log \Delta)} \) with distortion \( O(\log n \cdot (\log \Delta)^{1/p}) \). Note that for \( p = \infty \) the distortion is \( O(\log n) \).

Following the analysis of Bartal [2], we also obtain the following result.

**Theorem 2.** There exists an on-line algorithm that against a non-adaptive adversary, computes a probabilistic embedding of a given metric into a distribution over ultrametrics with distortion \( O(\log n \cdot \log \Delta) \).

We remark that in the off-line probabilistic embedding into ultrametrics of [2] the distortion is \( O(\log^2 n) \). In this bound there is no dependence on \( \Delta \) due to a preprocessing step that contracts all sufficiently small edges. This step however cannot be implemented in an on-line fashion, so the distortion bound in Theorem 2 is slightly weaker. Interestingly, Theorem 5 implies that removing the dependence on \( \Delta \) is impossible, unless the distortion becomes polynomially large.

### 3 Embedding doubling metrics into ultrametrics and into \( \ell_2 \)

In this section we give an embedding of doubling metrics into \( \ell_2 \) with distortion \( O(\log \Delta) \). We proceed by first giving a probabilistic embedding into ultrametrics. Let \( M = (X, D) \) be a doubling metric, with doubling dimension \( \lambda = \log_2 \Lambda \).

We begin with an informal description of our approach. Our algorithm proceeds by incrementally constructing an HST and embedding the points of the input space \( M \) into its leaves. The algorithm constructs an HST incrementally, embedding \( X \) into its leaves. The construction is essentially greedy: assume a good HST was constructed to the points so far, then when a new point \( p \) arrives it is necessary to “go down the right branch” of the tree so as to be at a small tree-distance away from points close to \( p \). This is done by letting each internal vertex of the HST of height \( i \) correspond to a subset of \( M \) of (appropriately randomized) radius about \( 2^i \). When \( p \) is too far from the previous centers of the balls it will branch out. The only issue that can arise (and in general, the only reason for randomness) is that while \( p \) is too far from the centre of a ball, it is in fact close to some of its members, and so a large expansion may occur when it is not placed in that part of the tree. Randomness allows to deal with this, but when decisions are made online and cannot be changed as in our case, it is not guaranteed to work. What saves the day is the fact that when a metric has bounded doubling dimension the obtained tree has bounded degree. This is crucial when bounding the probability of the bad event described above to happen, as at every level of the tree there could be only constant number of possible conflicts, each with low probability.

We now give a formal argument. Let \( \delta = \Lambda^3 \). Let \( T = (V, E) \) be a complete \( \delta \)-ary tree of depth \( \log \Delta \), rooted at a vertex \( r \). For each \( v \in V(T) \), let \( l(v) \) be the number of edges on the path from \( r \) to \( v \) in \( T \). We set the length of an edge \( \{u, v\} \in E(T) \) to \( \Delta \cdot 2^{-\min\{l(u), l(v)\}} \). That is, the length of the edges along a branch from \( r \) to a leaf, are \( \Delta, \Delta/2, \Delta/4, \ldots, 1 \). Fix a left-to-right orientation of the children of each vertex in \( T \). For a vertex \( v \in V(T) \), let \( T_v \) denote the sub-tree of \( T \) rooted at \( v \), and let \( c(v) \) denote the left-most leaf of \( T_v \). We refer to the point mapped to \( c(v) \) as the center of \( T_v \). Let \( B(x, r) \) denote the ball centered at \( x \) with radius \( r \).

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3See [3] Definition 8 for a definition of HST.
We will describe an on-line embedding $f$ of $M$ into $T$, against a non-adaptive adversary. We will inductively define mappings $f_1, f_2, \ldots, f_n = f$, with $f_i : \{x_1, \ldots, x_i\} \to V(T)$, such that $f_{i+1}$ is an extension of $f_i$. We pick a value $\alpha \in [1, 2]$, uniformly at random.

We inductively maintain the following three invariants.

(I1) For any $v \in V(T)$, if a point of $X_i$ is mapped to the subtree $T_v$, then there is a point of $X_i$ that is mapped to $c(v)$. In other words, the first point of $X$ that is mapped to a subtree $T_v$ has image $c(v)$, and is therefore the center of $T_v$. Formally, if $f_i(x_i) \cap V(T_v) \neq \emptyset$, then $c(v) \in f_i(X_i)$.

(I2) For any $v \in V(T)$, all the points in $X_i$ that are mapped to $T_v$ are contained inside a ball of radius $\Delta/2^{l(v)} - 1$ around the center of $T_v$ in $M$. Formally, $f_i^{-1}(V(T_v)) \subset B(f_i^{-1}(c(v)), \Delta/2^{l(v)} - 1)$.

(I3) For any $v \in V(T)$, and for any children $u_1 \neq u_2$ of $v$, the centers of $T_{u_1}$ and $T_{u_2}$ are at distance at least $\Delta/2^{l(v)} + 1$ in $M$. Formally, $D(f_i^{-1}(c(u_1)), f_i^{-1}(c(u_2))) > \Delta/2^{l(v)} + 1$.

We begin by setting $f_1(x_1) = c(r)$. This choice clearly satisfies invariants (I1)–(I3). Upon receiving a point $x_i$, we will show how to extend $f_{i-1}$ to $f_i$. Let $P = p_0, \ldots, p_l$ be the following path in $T$. We have $p_0 = r$. For each $j \geq 0$, if there exists a child $q$ of $p_j$ such that $V(q) \cap f_{i-1}(X_{i-1}) \neq \emptyset$, and $D(f_{i-1}^{-1}(c(q)), x_i) < \alpha \cdot \Delta/2^j$, we set $p_{j+1}$ to be the left-most such child of $p_j$. Otherwise, we terminate $P$ at $p_j$.

**Claim 1.** There exists a child $u$ of $p_l$, such that $c(u) \notin f_{i-1}(\{x_1, \ldots, x_{i-1}\})$.

**Proof.** Suppose that the assertion is not true. Let $y = f_{i-1}^{-1}(p_l)$. Let $v_1, \ldots, v_\delta$ be the children of $p_l$. By the inductive invariants (I1) and (I2), it follows that for each $i \in [\delta]$, $D(f_{i-1}^{-1}(c(v_i)), y) \leq \Delta/2^{l-1}$. Moreover, by the choice of $p_l$, $D(y, x_i) \leq \alpha \cdot \Delta/2^{l-1} - \Delta/2^{l-2}$. Therefore, the ball of radius $\Delta/2^{l-2}$ around $z$ in $M$, contains the $\delta + 1 = \Lambda^3 + 1$ points $x_i, f_{i-1}^{-1}(c(v_1)), \ldots, f_{i-1}^{-1}(c(v_\delta))$. However, by the choice of $p_l$, and by the inductive invariant (I3), it follows that the balls in $M$ of radius $\Delta/2^{l+1}$ around each one of these points are pairwise disjoint, contradicting the fact that the doubling constant of $M$ is $\Lambda$. \hfill \Box

By Claim 1 we can find a sub-tree rooted at a child $q$ of $p_l$ such that none of the points in $X_{i-1}$ has its image in $T_q$. We extend $f_{i-1}$ to $f_i$ by setting $f_i(x_i) = c(q)$. It is straight-forward to verify that $f_i$ satisfies the invariants (I1)–(I3). This concludes the description of the embedding. It remains to bound the distortion of $f$.

**Lemma 2.** For any $x, y \in X$, $D_T(f(x), f(y)) \geq \frac{1}{3} D(x, y)$.

**Proof.** Let $v$ be the nearest-common ancestor of $f(x)$ and $f(y)$ in $T$. By invariant (I2) we have $D(x, y) \leq D(x, f^{-1}(c(v))) + D(y, f^{-1}(c(v))) \leq \Delta \cdot 2^{-l(v)+2}$. Moreover, $D_T(f(x), f(y)) = 2 \cdot \Delta \sum_{i=l(v)} \Delta \cdot 2^{-i} = \Delta \cdot 2^{-l(v)+2} - 2$. The lemma follows since the minimum distance in $M$ is 1. \hfill \Box

**Lemma 3.** For any $x, y \in X$, $E[D_T(f(x), f(y))] \leq O(\Lambda^3 \cdot \log \Delta) \cdot D(x, y)$.

**Proof sketch (full details in the Appendix):** The distance between $x$ and $y$ is about $2^i$ when they are separated at level $i$. For this to happen, $y$ must be assigned to a sibling of $x$ at level $i - 1$. The probability of assigning to any particular such sibling is $O(D(x, y)/2^i)$. It is here that we utilize the bounded-degree property. By a union bound over all siblings at this level we get a contribution of $O(\Lambda^3)$ on the expected expansion. Summing up over all $\log \Delta$ levels we get the desired bound. \hfill \Box
Theorem 3. There exists an on-line algorithm that against any non-adaptive adversary, given a metric $M = (X, D)$ of doubling dimension $\lambda$, computes a probabilistic embedding of $M$ into a distribution over ultrametrics with distortion $2^{O(\lambda)} \cdot \log \Delta$.

It is well known that ultrametrics embed isometrically into $\ell_2$, and it is easy to see that such an embedding can be computed in an on-line fashion for the HSTs constructed above. We therefore also obtain the following result.

Theorem 4. There exists an on-line algorithm that against any non-adaptive adversary, given a doubling metric $M = (X, D)$ of doubling dimension $\lambda$, computes a probabilistic embedding of $M$ into $\ell_2$ with distortion $2^{O(\lambda)} \cdot \log \Delta$.

Remark 2. In the off-line setting, Krauthgamer et al. [15] have obtained embeddings of doubling metrics into Hilbert space with distortion $O(\sqrt{\log n})$. Their approach however is based on the random partitioning scheme of Calinescu, Karloff, and Rabani [5], and it is not known how to perform this step in the on-line setting.

4 Lower bound for probabilistic embeddings into ultrametrics

In this section we present a lower bound for on-line probabilistic embeddings into ultrametrics. Consider the following distribution $\mathcal{F}$ over metric spaces. Each space $M = (X, D)$ in the support of $\mathcal{F}$ is induced by an $n$-point subset of $\mathbb{R}^1$, with $X = \{x_1, \ldots, x_n\}$, and $D(x_i, x_j) = |x_i - x_j|$. We have $x_1 = 0$, $x_2 = 1$, $x_3 = 1/2$. For each $i \geq 4$, we set $x_i = x_{i-1} + b_i 2^{-i+2}$, where $b_i \in \{-1, 1\}$ is chosen uniformly at random.

![Figure 1: The evolution of the construction of the ultrametric.](image)

It is easy to see that for each $i \geq 3$, there exist points $l_i, r_i \in X$ such that $l_i = x_i - 2^{-i+2}$, $r_i = x_i + 2^{-i+2}$, and $\{x_1, \ldots, x_i\} \cap [l_i, r_i] = \{l_i, x_i, r_i\}$. Moreover, for each $i \in \{3, \ldots, n-1\}$, there uniquely exists $y_i \in \{l_i, r_i\}$, such that $\{x_{i+1}, \ldots, x_n\} \subset [\min\{x_i, y_i\}, \max\{x_i, y_i\}]$.

Claim 2. Let $M = (X, D)$ be a metric from the support of $\mathcal{F}$. Let $f$ be an embedding of $M$ into an ultrametric $M' = (X, D')$. Then, for each $i \geq 3$, there exists $z_i \in \{l_i, r_i\}$, such that $D'(x_i, z_i) \geq D'(l_i, r_i)$.

Proof. It follows immediately by the fact that $M'$ is an ultrametric, since for any $x_i, l_i, r_i \in X$, $D'(l_i, r_i) \leq \max\{D'(l_i, x_i), D'(x_i, r_i)\}$. □

In order to simplify notation, we define for any $i \geq 4$, $\delta_i = D(x_i, y_i)$, and $\delta'_i = D'(x_i, y_i)$.

Claim 3. Let $M = (X, D)$ be a metric from the support of $\mathcal{F}$. Let $f$ be an on-line embedding of $M$ into an ultrametric $M' = (X, D')$. Then, for any $i \geq 3$, $\Pr[\delta'_i \geq \delta'_{i-1} \mid \forall j \in \{4, \ldots, i-1\}, \delta'_j \geq \delta'_{j-1}] \geq 1/2$.

Proof. Assume without loss of generality that $z_i = l_i$, since the case $z_i = r_i$ is symmetric. By the construction of $M$, we have that $\Pr[y_i = z_i \mid \forall j \in \{4, \ldots, i-1\}, \delta'_j \geq \delta'_{j-1}] = 1/2$. If $y_i = z_i$, then $\delta'_i = D'(x_i, z_i) \geq D'(l_i, r_i) = \delta'_{i-1}$, concluding the proof. □
Lemma 4. Let \( f \) be an on-line, non-contracting embedding of \( M \) into an ultrametric \( M' \). Then, \( \mathbb{E}[\delta'_{n-1}/\delta_{n-1}] = \Omega(n) \).

**Proof.** Let \( i \geq 4 \), and \( 1 \leq t \leq i - 1 \). By Claim 3 we have \( \mathbb{P}[\delta'_i \geq \delta_{i-1}] \geq \mathbb{P}[\delta'_j \geq \delta_{i-1}] \geq \mathbb{P}[\forall j \in \{1, \ldots, t\}, \delta'_{i-j+1} \geq \delta'_{i-j}] = \prod_{j=1}^{t} \mathbb{P}[\delta'_{i-j+1} \geq \delta'_{i-j}] \). Therefore \( \mathbb{E}[\delta'_{n-1}] \geq \sum_{i=3}^{t} \sum_{k=1}^{n-1} 2^{-i+k} \geq 2^{-i} \cdot 2^{-n+i+1} = \Omega(n \cdot 2^{-n}) = \Omega(n) \cdot \delta_{n-1} \). \[ \square \]

Since the aspect ratio (spread) is \( \Delta = \Theta(2^n) \), we obtain the following result.

**Theorem 5.** There exists a non-adaptive adversary against which any on-line probabilistic embedding into a distribution over ultrametrics has distortion \( \Omega(\min\{n, \log \Delta\}) \).

We remark that the above bound is essentially tight, since the input space is a subset of the line, and therefore doubling. By Theorem 3, every doubling metric space probabilistically embeds into ultrametrics with distortion \( O(\log \Delta) \).

5 Embedding into \( \ell_\infty \)

In the off-line setting, it is well-known that any \( n \)-point metric space isometrically embeds into \( n \)-dimensional \( \ell_\infty \). Moreover, there is an explicit construction of the embedding due to Fréchet. Let \( M = (X, D) \) be an arbitrary metric space. The embedding \( f : (X, D) \to \ell_\infty \) is simply \( f(x_i) = (D(x_i, x_1), D(x_i, x_2), \ldots, D(x_i, x_n)) \). It is clear that the Fréchet embedding does not fit in the on-line setting, since the image of any point \( x \) depends on the distances between \( x \) and all points of the metric space, in particular the future points.

A similar question regarding the existence of on-line embeddings can be posed: does there exist a bi-Lipschitz extension for any embedding into \( \ell_\infty \)? The connection with the on-line setting is immediate; it is well-known (see e.g. [16]) that for any metric space \( M = (X, D) \), for any \( Y \subseteq X \), and for any \( a \)-Lipschitz function \( f : Y \to \ell_\infty \), there exists an \( a \)-Lipschitz extension \( \tilde{f} \) of \( f \), with \( \tilde{f} : X \to \ell_\infty \). It seems natural to ask whether this is also true when \( f \) and \( \tilde{f} \) are required to be \( a \)-bi-Lipschitz. Combined with the fact that any metric embeds isometrically into \( \ell_\infty \), this would immediately imply an on-line algorithm for embedding isometrically into \( \ell_\infty \): start with an arbitrary isometry, and extend it at each step to include a new point. Unfortunately, as the next proposition explains, this is not always possible, even for the special case of \( (1, 2) \)-metrics (the proof appears in the Appendix). We need some new ideas to obtain such an embedding.

**Proposition 1.** There exists a finite metric space \( M = (X, D) \), \( Y \subset X \), and an isometry \( f : Y \to \ell_\infty \), such that any extension \( \tilde{f} : X \to \ell_\infty \) of \( f \) is not an isometry.

Although it is not possible to extend any \( 1 \)-bi-Lipschitz mapping into \( \ell_\infty \), there exists a specific mapping that is extendable, provided that the input space is drawn from a fixed finite family of metrics. We will briefly sketch the proof of this fact, and defer the formal analysis to the Appendix. Consider a metric space \( M' \) obtained from \( M \) by adding a point \( p \). Suppose that we have an isometry \( f : M \to \ell_\infty \). As explained above, \( f \) might not be isometrically extendable to \( M' \). The key step is proving that \( f \) is always Lipschitz-extendable to \( M' \). We can therefore get an on-line embedding as follows: We maintain a concatenation of embeddings for all metrics in the family of input spaces. When we receive a new point \( x_i \), we isometrically extend all embeddings of spaces that agree with our input on \( \{x_1, \ldots, x_i\} \), and Lipschitz-extend the rest.

**Theorem 6.** Let \( \mathcal{F} \) be a finite collection of \( n \)-point metric spaces. There exists an on-line embedding algorithm that given a metric \( M \in \mathcal{F} \), computes an isometric embedding of \( M \) into \( \ell_\infty \).
5.1 Low-distortion embeddings into low-dimensional $\ell_\infty$

In the pursuit of a good embedding of a general metric space into low dimensional $\ell_\infty$ space we demonstrate the usefulness (and feasibility) of concatenation of two on-line embeddings. In fact one of these embeddings is oblivious, which in particular makes it on-line. Why the concatenation of two on-line embeddings results in yet another on-line embeddings is fairly clear when the embeddings are deterministic; in the case of probabilistic embeddings it suffices to simply concatenate the embeddings in an independent way. In both cases the distortion is the product of the distortions of the individual embeddings. Recall that Section 2 provides us with an on-line embedding of a metric space into Euclidean space. The rest of the section shows that the classical method of projection of points in Euclidean space onto a small number of dimensions supplies low distortion embedding when the host space is taken to be $\ell_\infty$. To put things in perspective, the classical Johnson-Lindenstrauss lemma [13] considers the case where the image space is equipped with the $\ell_2$ norm, and it is well-known that a similar result can be achieved with $\ell_1$ as the image space [11, p. 92]. As we will see, $\ell_\infty$ metric spaces behave quite differently than $\ell_2$ and $\ell_1$ spaces in this respect, and while a dimension reduction is possible, it is far more limited than the first two spaces.

The main technical ingredient we need is the following concentration result. See also [23] for a similar analysis. The proof is given in the Appendix.

**Lemma 5.** Let $u \in \mathbb{R}^n$ be a nonzero vector and let $\alpha > 1$ and $d \geq e^2$. Let $y$ be the normalized projection of $u$ onto $d$ dimensions by a Gaussian matrix as follows: $y = (2/m)Ru$ where $R$ is a $d \times n$ Gaussian random matrix, i.e., a matrix with i.i.d. normal entries and $m = 2\sqrt{\ln d}$. Then $\Pr \left[ \|y\|_\infty / \|u\|_2 \leq 1 \right] \leq \exp(-\frac{1}{4}\frac{d}{\ln d})$, and $\Pr \left[ \|y\|_\infty / \|u\|_2 \geq \alpha \right] \leq (2/\alpha)d^{1-\alpha^2/2}$.

With the concentration bound of Lemma 5 it is not hard to derive a good embedding for any $n$-point set, as is done, say, in the Johnson Lindenstrauss Lemma [13], and we get

**Lemma 6.** Let $X \subset \mathbb{R}^n$ an $n$-point set and let $\alpha > \sqrt{2}$. If $d = \Omega(\max\{\log n, n^{4/(\alpha^2 - 2)}\})$, then the above mapping $f : X \rightarrow \ell_\infty^d$ satisfies $\forall x, y \in X, \|x - y\|_2 \leq \|f(x) - f(y)\|_\infty \leq \alpha \|x - y\|_2$ with high probability.

By a straightforward composition of the embeddings in Theorem 1 and Lemma 6 we get

**Theorem 7.** There exists an on-line algorithm against any non-adaptive adversary that for any $\alpha > \sqrt{2}$, given a metric $M = (X, D_M)$, computes an embedding of $M$ into $\ell_\infty^d$ with distortion $O(\alpha \cdot \log n \cdot \sqrt{\log \Delta})$ and $d = \Omega(\max\{\log n, n^{4/(\alpha^2 - 2)}\})$.

**Remark 3.** The embeddings into $\ell_\infty$ given in Theorems 1 and 7 are incomparable: the distortion in Theorem 7 is smaller, but the dimension is larger than the one in Theorem 1 for large values of $\Delta$.

6 On-line embedding when an off-line (near-)isometry is possible

It is not hard to see that given an $n$-point $\ell_2^d$ metric $M$, one can compute an online isometric embedding of $M$ into $\ell_2^d$. This is simply because there is essentially (up to translations and rotations) a unique isometry, and so keeping extending the isometry online is always possible. However, as soon as we deal with near isometries this uniqueness is lost, and the situation changes dramatically as we next show: even when the input space embeds into $\ell_2^d$ with distortion $1 + \varepsilon$, the best online embedding we can guarantee in general will have distortion that is polynomial in $n$. We use the following topological lemma from [20].

**Lemma 7 ([20]).** Let $\delta < \frac{1}{4}$ and let $f_1, f_2 : S^{d-1} \rightarrow \mathbb{R}^d$ be continuous maps satisfying
\[ \|f_i(x) - f_i(y)\|_2 \geq \|x - y\|_2 - \delta \text{ for all } x, y \in S^{d-1} \text{ and all } i \in \{1, 2\}, \]

\[ \|f_1(x) - f_2(x)\|_2 \leq \frac{1}{4} \text{ for all } x \in S^{d-1}, \text{ and} \]

\[ \Sigma_1 \cap \Sigma_2 = \emptyset, \text{ where } \Sigma_i = f_i(S^{d-1}). \]

Let \( U_i \) denote the unbounded component of \( \mathbb{R}^d \setminus \Sigma_i. \) Then, either \( U_1 \subset U_2, \) or \( U_2 \subset U_1. \)

**Theorem 8.** For any \( d \geq 2, \) for any \( \varepsilon > 0, \) and for sufficiently large \( n > 0, \) there exists a distribution \( \mathcal{F} \) over \( n\)-point metric spaces that embed into \( \ell_2^n \) with distortion \( 1 + \varepsilon, \) such that any on-line algorithm on input a metric space chosen from \( \mathcal{F} \) outputs an embedding into \( \ell_2^n \) with distortion \( \Omega(n^{1/(d-1)}), \) and with probability at least \( 1/2. \)

**Proof.** Let \( \gamma = 1/(\alpha \cdot n^{1/(d-1)}), \) where \( \alpha > 0 \) is a sufficiently large constant. Let \( S^{d-1} \) denote the unit \((d - 1)\)-dimensional sphere, and let \( X \) be a \( \gamma \)-net of \((S^{d-1}, \| \cdot \|_2). \) That is, for any \( x_1, x_2 \in X, \) we have \( \|x_1 - x_2\|_2 > \gamma, \) and for any \( y \in S^{d-1} \) there exists \( x \in X \) with \( \|x - y\|_2 \leq \gamma. \) Such a set \( X \) can be always constructed with \( |X| \leq O((1/\gamma)^{d-1}). \) We will assume for convenience (and without loss of generality) that \( X \) contains the point \((1, 0, \ldots, 0). \)

![Figure 2: A realization of \( Y \) in \( \mathbb{R}^3 \) for \( d = 2, \) and a \((1 + \varepsilon)\)-embedding into \( \mathbb{R}^2.\)](image)

Let \( t \in \{1, 2\}. \) We will define a metric space \( M_t = (Y, D_t) \) containing two copies of \( X, \) and a discretization of a line segment. Formally, we have \( Y = (\{1, 2\} \times X) \cup Z, \) where \( Z = \{z_1, \ldots, z_1/\gamma^2\}. \) The distance \( D_t : Y \times Y \to \mathbb{R}_{\geq 0} \) is defined as follows. For each \( i \in \{1, 2\}, \) the set \( i \times X \) induces a metric copy of \( X, \) as a subset of \( \mathbb{R}^d. \) That is, for any \( p, q \in X \) we have \( D_t((i, p), (i, q)) = \|p - q\|_2. \) Moreover, for any \( p, q \in X \) we have \( D_t((1, p), (2, q)) = \sqrt{\varepsilon^2 + \|p - q\|_2^2}. \) The distance \( D_t \) induces on \( Z \) a line metric. That is, for any \( z_j, z_k \in Z \) we have \( D_t(z_j, z_k) = |j - k| \cdot \gamma. \) For any \( z_j \in Z, \) and for any \( p \in X \) \( D_t(z_j, (t, p)) = \|p - (1 + j \cdot \gamma, 0, 0, \ldots, 0)\|_2. \) and \( D_t(z_j, (3 - t, p)) = \sqrt{\varepsilon^2 + \|p - (1 + j \cdot \gamma, 0, 0, \ldots, 0)\|_2^2}. \)

We first argue that for every \( t \in \{1, 2\}, \) the metric space \( M_t \) embeds into \( \mathbb{R}^d \) with distortion \( 1 + \varepsilon. \) To see that, consider the embedding \( g : Y \to \mathbb{R}^d, \) where for all \( p \in X, g((t, p)) = (1 + \varepsilon) \cdot (3 - t, p) = p, \) and \( g(z_i) = (1 + \varepsilon + i \cdot \gamma, 0, \ldots, 0). \) It is straight-forward to check that the distortion of \( g \) is \( 1 + \varepsilon. \)

We define \( \mathcal{F} \) to be the uniform distribution over \( \{M_1, M_2\}. \) It remains to show that any on-line algorithm, on input a space \( M_t \) chosen from \( \mathcal{F}, \) outputs an embedding with distortion at least \( \Omega(n^{1/(d-1)}), \) and with probability at least \( 1/2. \) We assume that the on-line algorithm receives first in its input the points in \( \{1, 2\} \times X. \) Let \( f \) be the embedding computed by the algorithm. We can assume without loss of generality that \( f \) is non-contracting, and that it is \( c \)-Lipschitz, for some \( c \geq 1. \) For any \( i \in \{1, 2\} \) let \( f_i \) be the restriction of \( f \) on \( \{i\} \times X. \) By Kirszbraun’s theorem \([14], \) each \( f_i \) can be extended to a continuous map \( \tilde{f}_i : S^{d-1} \to \mathbb{R}^d, \) that is also \( c \)-Lipschitz. If \( c > n^{1/(d-1)} \), then there is nothing to prove, so we may assume \( c \leq n^{1/(d-1)}. \)

It follows by the analysis in \([20] \) that (i) \( \tilde{f}_1(S^{d-1}) \cap \tilde{f}_2(S^{d-1}) = \emptyset, \) (ii) for any \( x, y \in S^{d-1} \) \( i \in \{1, 2\}, \) we have \( \|f_i(x) - f_i(y)\|_2 \geq \|x - y\|_2 - O(c \cdot \gamma), \) and (iii) for any \( x \in S^{d-1}, \) we have \( \|f_1(x) - f_2(x)\|_2 = \)
Therefore, we can apply Lemma on \( \bar{f}_1 \) and \( \bar{f}_2 \). For each \( i \in \{1, 2\} \), let \( U_i = \mathbb{R}^d \setminus \bar{f}_i(S^{d-1}) \). It follows that either \( U_1 \subset U_2 \), or \( U_2 \subset U_1 \).

Observe that since the algorithm receives first \( \{1, 2\} \cup X \), and \( M_t \) is symmetric on \( \{1\} \times X \), \( \{2\} \times X \), it follows that \( \Pr_{M_t \in F} [U_{3-t} \subset U_t] = 1/2 \). Let \( p_0 = (1, 0, \ldots, 0) \). Observe that \( \bar{f}_{3-t}(S^{d-1}) \subseteq B(\bar{f}_{3-t}(p_0), c) \), and on the other hand \( \|f(z_{1/\gamma}) - \bar{f}_{3-t}(p_0)\|_2 > 1/\gamma > c \). We thus obtain \( f(z_{1/\gamma}) \in U_{3-t} \).

Let \( \bar{Z} \) be the polygonal curve in \( \mathbb{R}^d \) obtained via affine extension of \( f \) on \( p_0, z_1, \ldots, z_{1/\gamma^2} \). It follows that \( \Pr_{M_t \in F} [\bar{Z} \cap \bar{f}_{3-t}(S^{d-1}) \neq \emptyset] \geq 1/2 \). Therefore, with probability at least \( 1/2 \), two points in \( Y \) that are at distance \( \Omega(\varepsilon) \) in \( M_t \), are mapped to points that are at distance \( O(\gamma c) \). Thus \( c = \Omega(1/\gamma) = \Omega(n^{1/(d-1)}) \). \( \square \)

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References


Proof. Let $X = \{x_1, x_2, x_3, x_4\}$. Let $D(x_2, x_1) = D(x_2, x_3) = D(x_2, x_4) = 1$, and let the distance between any other pair of points be 2. Let $Y = \{x_1, x_2, x_3\}$. Consider an isometry $f : Y \to \ell_\infty$, were for each $i \in \{1, 2, 3\}$, $f_1(x_i) = i$, and for each $j > 1$, $f_j(x_i) = 0$.

Let now $\tilde{f} : X \to \ell_\infty$ be an extension of $f$. Assume for the sake of contradiction that $\tilde{f}$ is 1-bi-Lipschitz. Then, $\tilde{f}_1(x_4)$ must lie between $\tilde{f}_1(x_1)$ and $\tilde{f}_1(x_2)$, since otherwise either $|\tilde{f}_1(x_4) - \tilde{f}_1(x_1)| > 1$, or $|\tilde{f}_1(x_2) - \tilde{f}_1(x_1)| > 1$. Similarly, $\tilde{f}_1(x_4)$ must also lie between $\tilde{f}_1(x_2)$ and $\tilde{f}_1(x_3)$. Thus, $\tilde{f}_1(x_4) = 2$.

Since $\tilde{f}$ is 1-bi-Lipschitz, it follows that there exists a coordinate $j$, such that $|\tilde{f}_j(x_4) - \tilde{f}_j(x_1)| = 2$. Since $|\tilde{f}_1(x_4) - \tilde{f}_1(x_1)| = 1$, it follows that $j > 1$. Thus, $|\tilde{f}_j(x_4) - \tilde{f}_j(x_2)| = |\tilde{f}_j(x_4) - \tilde{f}_j(x_1)| = 2 > D(x_4, x_1)$, contradicting the fact that $\tilde{f}$ is 1-bi-Lipschitz. 

$\Box$
Proof of Theorem 3

Proof. Let \( P^x = p^x_0, \ldots, p^x_{\log \Delta} \), and \( P^y = p^y_0, \ldots, p^y_{\log \Delta} \) be paths in \( T \) between \( p^x_0 = p^y_0 = r \) and \( p^x_{\log \Delta} = f(x), p^y_{\log \Delta} = f(y) \) respectively. Assume that \( x \) appears before \( y \) in the sequence \( x_1, \ldots, x_n \). For any \( i = 0, \ldots, \log \Delta - 1 \) we have

\[
\Pr[p^x_{i+1} \neq p^y_{i+1} | p^x_i = p^y_i ] \leq \sum_{v \text{ child of } p^x_i} \Pr[p^x_{i+1} = v \text{ and } p^y_{i+1} \neq v | p^x_i = p^y_i ]
\]

\[
\leq \sum_{v \text{ child of } p^x_i} \Pr[D(x, f^{-1}(c(v))) \leq \alpha \cdot \Delta/2^{i+1} < D(y, f^{-1}(c(v)))]
\]

\[
\leq \sum_{v \text{ child of } p^x_i} \Pr \left[ \alpha \in \left[ \frac{2^{i+1}}{\Delta} D(x, f^{-1}(c(v))), \frac{2^{i+1}}{\Delta} D(y, f^{-1}(c(v))) \right] \right]
\]

\[
\leq \sum_{v \text{ child of } p^x_i} \frac{2^{i+1}}{\Delta} |D(x, f^{-1}(c(v))) - D(y, f^{-1}(c(v)))|
\]

\[
= \delta \cdot \frac{2^{i+1}}{\Delta} \cdot D(x, y).
\]

When the nearest common ancestor (nca) of \( f(x) \) and \( f(y) \) in \( T \) is \( p^x_0 \), we have \( D_T(f(x), f(y)) = \Delta \cdot 2^{-i+2} - 2 \). Therefore

\[
\mathbb{E}[D_T(f(x), f(y))] = \sum_{i=0}^{\log \Delta - 1} \Pr[\text{nca}(f(x), f(y)) = p^x_i] \cdot (\Delta \cdot 2^{-i+2} - 2)
\]

\[
< \sum_{i=0}^{\log \Delta - 1} \Pr[p^x_i = p^y_i \text{ and } p^x_{i+1} \neq p^y_{i+1}] \cdot \Delta \cdot 2^{-i+2}
\]

\[
\leq \sum_{i=0}^{\log \Delta - 1} \delta \cdot \frac{2^{i+1}}{\Delta} \cdot D(x, y) \cdot \Delta \cdot 2^{-i+2}
\]

\[
= O(\Lambda^2 \cdot \log \Delta) \cdot D(x, y)
\]

\[\square\]

### 6.1 Proof of Theorem 6

We give a formal analysis of the on-line algorithm for embedding isometrically into \( \ell_\infty \). Let \( M = (X, D) \) be the input metric, with \( X = \{x_1, \ldots, x_n\} \). Let \( M_i \) denote the restriction of \( M \) on \( \{x_1, \ldots, x_i\} \).

The following Lemma is the well-known Helly property of intervals of \( \mathbb{R} \) (see e.g. [19]).

**Lemma 8** (Helly property of line intervals). Consider a finite collection of closed real intervals \( \Lambda_1, \ldots, \Lambda_k \). If for any \( i, j \in [k] \), \( \Lambda_i \cap \Lambda_j \neq \emptyset \), then \( \bigcap_{i=1}^{k} \Lambda_i \neq \emptyset \).
The main idea of the algorithm is as follows. Each coordinate of the resulting embedding is clearly a line metric. The algorithm will produce an embedding that inductively satisfies the property that at each step $i$, for each possible line metric $T$ dominated by $M_i$, there exists a coordinate inducing $M_i$.

The main problem is what to do with the line metrics that are dominated by $M_i$, but are not dominated by $M_{i+1}$. As the next lemma explains, we can essentially replace such metrics by line metrics that are dominated by $M_{i+1}$.

Lemma 9. Let $T$ be a line metric on $\{x_1, \ldots, x_i\}$ dominated by $M_i$. Then, $T$ can be extended to a line metric $T'$ on $\{x_1, \ldots, x_{i+1}\}$ dominated by $M_{i+1}$.

Proof. Consider $b_1, \ldots, b_i \in \mathbb{R}$, such that for any $j, k \in [i]$, $D_T(x_j, x_k) = |b_j - b_k|$. For any $j \in [i]$, define the interval $\Lambda_j = [b_j - D(x_j, x_{i+1}), b_j + D(x_j, x_{i+1})]$. Assume now that there exist $j, k \in [i]$, such that $\Lambda_j \cap \Lambda_k = \emptyset$. Assume without loss of generality that $b_j \leq b_k$. It follows that $b_j + D(x_j, x_{i+1}) < b_k - D(x_k, x_{i+1})$. Since $D(x_j, x_k) = b_k - b_j$, it follows that $D(x_j, x_{i+1}) + D(x_{i+1}, x_k) < D(x_j, x_k)$, contradicting the triangle inequality.

We have thus shown that for any $j, k \in [i]$, $\Lambda_j \cap \Lambda_k \neq \emptyset$. By Lemma 8 it follows that there exists $b_{i+1} \in \bigcap_{j=1}^i \Lambda_j$. We can now define the extension $T'$ of $T$ by $D_{T'}(x_j, x_k) = |b_j - b_k|$. It remains to verify that $T'$ is dominated by $M_{i+1}$. Since $T$ is dominated by $M_i$, it suffices to consider pairs of points $x_j, x_{i+1}$, for $j \in [i]$. Since $b_{i+1} \in \Lambda_j$, we have $D_{T'}(x_j, x_{i+1}) = |b_j - b_{i+1}| \leq D(x_j, x_{i+1})$, which concludes the proof.

Proof of Lemma 5

Proof. As is standard, we will use homogeneity and the rotation invariance of the operator $R$ to assume without loss of generality that $u = (1, 0, \ldots, 0)^t$. It follows that $\|y\|_\infty = (2/m)\|z\|_\infty$ where $z = (z_1, z_2, \ldots, z_d)^t$ and $z_i \sim \mathcal{N}(0, 1)$ for $i = 1, \ldots, d$. Denote by $\varphi(x)$ the probability density function of the normal distribution $\mathcal{N}(x^2/2, \sqrt{2\pi})$ and the probability distribution function $\Phi(x) = \int_{-\infty}^x \varphi(t) dt = \Pr[X \leq x]$.

---

Footnote: Here we mean that an $i$-point metric dominates an $n$-point metric, if it dominates its restriction on the first $i$ points.
where $X \sim \mathcal{N}(0, 1)$. We will use the well-known fact to estimate $\Phi$ (see [7, Lemma 2, p. 131])

$$\frac{x^2 - 1}{x^3} \varphi(x) \leq 1 - \Phi(x) \leq \frac{\varphi(x)}{x}. \quad (1)$$

For the left tail of the lemma, we get that

$$\Pr[\|y\|_\infty \leq 1] = \Pr[\|z\|_\infty \leq m/2] = \Pr[\bigcap_i \{z \mid |z_i| \leq m/2\}] \leq (\Phi(m/2))^d$$

$$\leq \exp \left( -d \frac{(2m^2 - 8)\varphi(m/2)}{m^3} \right) \quad (Eqn. 1)$$

$$\leq \exp \left( -d \frac{e^{-m^2/8}}{m \sqrt{2\pi}} \right) \leq \exp \left( -d \frac{d^{-1/2}}{4 \sqrt{\ln d}} \right)$$

$$= \exp \left( -\frac{1}{4} \sqrt{d/\ln d} \right).$$

We turn to bound the right tail.

$$\Pr[\|y\|_\infty \geq \alpha] = \Pr[\|z\|_\infty \geq m\alpha/2] = \Pr \left[ \bigcup_i \{z \mid |z_i| \geq m\alpha/2\} \right]$$

$$\leq 2d \cdot (1 - \Phi(m\alpha/2))$$

$$\leq 2d \frac{\varphi(m\alpha/2)}{m\alpha/2} = \frac{4d}{m\alpha \sqrt{2\pi}} e^{-m^2\alpha^2/8} \quad (Eqn. (1))$$

$$= \frac{2d}{\alpha \sqrt{2\pi \ln d}} d^{-\alpha^2/2} \leq (2/\alpha)d^{1-\alpha^2/2}.$$