6331 - Algorithms, Spring 2014, CSE, OSU
Lecture 2: Sorting

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Sorting

Given an array of integers $A[1 \ldots n]$, rearrange its elements so that

$$A[1] \leq A[2] \leq \ldots \leq A[n].$$

A simple sorting algorithm

Bubble-Sort

```
repeat
  swapped = false
  for i = 1 to n − 1 do
    if A[i − 1] > A[i] then
      swap(A[i − 1], A[i])
      swapped = true
    end if
  end for
until not swapped
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What is the worst-case time complexity of this algorithm?

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What is the worst-case time complexity of this algorithm?

We can do much better!

Heaps

A *Heap* is a data structure representing a full binary tree.

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- A heap is stored in an array $A[1 \dots n]$.
- The root is $A[1]$.
- $\text{parent}(i) = i/2$.
- $\text{left-child}(i) = 2i$.
- $\text{right-child}(i) = 2i + 1$.

Max-Heaps

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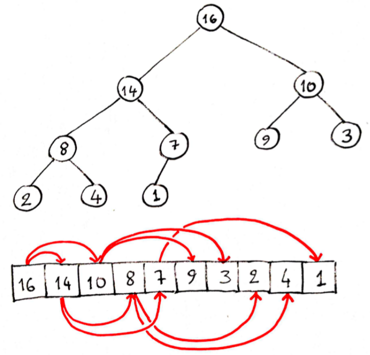

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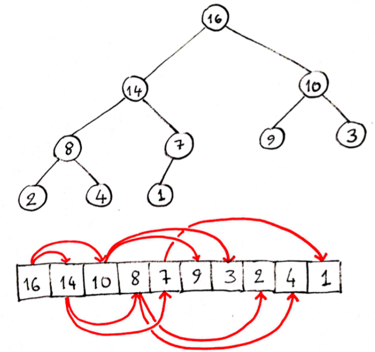

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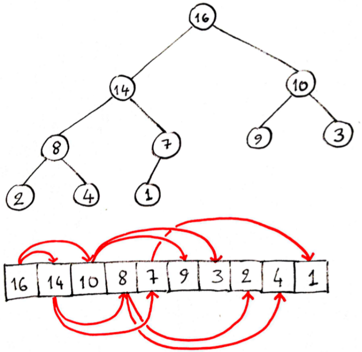

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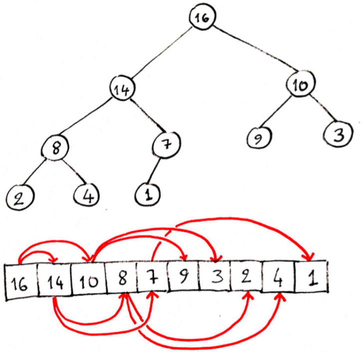

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- Where is the maximum element in the array?
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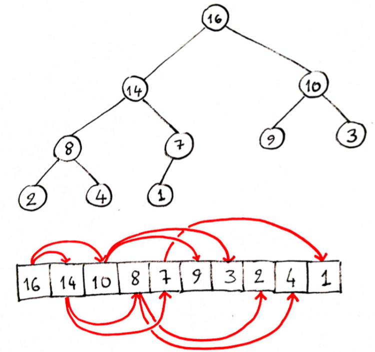

- Where is the maximum element in the tree?
- Where is the maximum element in the array?
- Where is the minimum element in the tree?
- Where is are the leaves in the array?

Height

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What is the height of a heap?

Building and using heaps

- Procedure Max-Heapify (auxiliary procedure)
- Procedure Build-Max-Heap (building a max-heap)
- Procedure Heap-Sort (sorting using a heap)

Maintaining the max-heap property

Suppose that the subtrees rooted at left-child(i) and right-child(i) are max-heaps.

However, i might violate the max-heap property.
E.g., $A[i] < A[\text{left-child}(i)]$.

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How can we enforce the max-heap property?

Maintaining the max-heap property

```
Procedure Max-Heapify(A, i)
  l = left-child(i)
  r = right-child(i)
  if l ≤ n and A[l] > A[i]
    largest = l
  else largest = i
  if r ≤ n and A[r] > largest
    largest = r
  if largest ≠ i
    exchange A[i] with A[largest]
    Max-Heapify(A, largest)
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- Worst-case depth of recursion $=$ height of i.
- Worst-case running time is $O(\text{height}(i))$.
- Worst-case running time is $O(\log(n))$.
- Is this tight?

Building a heap

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- **Maintenance:** By the loop invariant, the children of i are roots of max-heaps. Therefore, running Max-Heapify makes i the root of a max-heap.
- **Termination:** $i = 0$. By the loop invariant, 1 is the root of a heap.

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- This is not asymptotically tight!

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- Each call Max-Heapify(A, i) takes time $O(\text{height}(i))$.
- A heap has height $\lfloor \log(n) \rfloor$.
- There are at most $\lceil n/2^{h+1} \rceil$ nodes of height h.
- Total running time:

$$\sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$
$$= O(n).$$

Sorting using a heap

Procedure Heapsort(A)
  Build-Max-Heap(A)
  for $i = A$.length downto 2
    exchange $A[1]$ with $A[i]$
    A.heap-size $= A$.heap-size $- 1$
    Max-Heapify($A, 1$)

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- There are $n-1$ calls to Max-Heapify.
- Each call to Max-Heapify takes time $O(\log n)$.
- Total running time $O(n \log(n))$.

Priority queues

A *priority queue* is a data structure for maintaining a set S of elements, each having a *key*.

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Operations of a max-priority queue:

- Insert(S, x): $S = S \cup \{x\}$.
- Maximum(S): Return the element in S with the maximum key.
- Extract-Max(S): Removes and returns the element in S with the maximum key.
- Increase-Key(S, x, k): Increases the value of the key of x to k, assuming that k is larger than the current value.

Implementing a max-priority queue using a max-heap

Procedure Heap-Maximum(A)

Implementing a max-priority queue using a max-heap

Procedure Heap-Maximum(A)
  return $A[1]$

Implementing a max-priority queue using a max-heap

Procedure Heap-Extract-Max(A)

Implementing a max-priority queue using a max-heap

```
Procedure Heap-Extract-Max(A)
  if n < 1
    error "empty heap"
  max = A[1]
  A[1] = A[n]
  n = n − 1
  Max-Heapify(A, 1)
  return max
```

Implementing a max-priority queue using a max-heap

Procedure Heap-Increase-Key(A, i, key)

Implementing a max-priority queue using a max-heap

Procedure Heap-Increase-Key(A, i, key)
  if key $< A[i]$
    error
  $A[i] =$ key
  while $i > 1$ and $A[\text{parent}(i)] < A[i]$
    exchange $A[i]$ with $A[\text{parent}(i)]$
    $i = \text{parent}(i)$

Implementing a max-priority queue using a max-heap

Procedure Max-Heap-Insert(A, key)

Implementing a max-priority queue using a max-heap

Procedure Max-Heap-Insert(A, key)
  $n = n + 1$
  $A[n] = -\infty$
  Heap-Increase-Key(A, n, key)