6331 - Algorithms, Spring 2014, CSE, OSU
Elementary graph algorithms

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Graph problems

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- The running time is measured in terms of $|V|$, and $|E|$.

Representing a graph

Adjacency-matrix for a graph $G = (V, E)$.

$|V| \times |V|$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{if } \{i, j\} \notin E \end{cases}$$

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Much smaller space when $|E| \ll |V|^2$.

Breadth-first search

An algorithm for “exploring” a graph, starting from the given vertex s.

Breadth-first search

```
BFS(G, s)
    for each u ∈ G.V − {s}
        u.color = WHITE
        u.d = ∞
        u.π = NIL
    s.color = GRAY
    s.d = 0
    s.π = NIL
    Q = ∅
    ENQUEUE(Q, s)     //FIFO queue
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ G.Adj[u]
            if v.color = WHITE
                v.color = GRAY
                v.d = u.d + 1
                v.π = u
                ENQUEUE(Q, v)
        u.color = BLACK
```

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- Total running time $O(|V| + |E|)$.

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A path between u and v in G of length $\delta(u, v)$ is called a **shortest-path**.

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Lemma

For any $\{u, v\} \in E$, *we have*

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$$\begin{aligned} v.d &= u.d + 1 \\ &\geq \delta(s, u) + 1 \\ &\geq \delta(s, v) && \text{(by the previous Lemma)} \end{aligned}$$

$\square$

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Lemma

Suppose during the execution, $Q = (v_1, \ldots, v_r)$*, where* $v_1 = head$*,* $v_r = tail$*. Then for all* $i \in \{1, \ldots, r-1\}$

$$v_i.d \leq v_{i+1}.d,$$

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Suppose during the execution, both v_i and v_j are enqueued, and v_i is enqueued before v_j. Then, $v_i.d \leq v_j.d$ when v_j is enqueued.

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Theorem

After termination, for all $v \in V$, we have

$$v.d = \delta(s, v).$$

Moreover, for any v that is reachable from s, there exists a shortest path from s to v that consists of a shortest path from s to $v.\pi$, followed by the edge $\{v.\pi, v\}$.

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Let u be the vertex preceding v in a shortest path from s to v. We have

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- If v is WHITE, then $v.d = u.d + 1$, a contradiction.
- If v is BLACK, then it is already dequeued, so by the above Lemma $v.d \leq u.d$, a contradiction.
- If v is GRAY, then it was painted GRAY after dequeueing some vertex w, so $v.d = w.d + 1 \leq u.d + 1$, a contradiction.

Proof sketch (cont.)

So, $v.d = \delta(s, v)$ for all $v \in V$.

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For the last part of the theorem, if $u = v.\pi$, then $v.d = u.d + 1$. The assertion follows by induction.

Breadth-first trees

We define the **predecessor graph** as $G_\pi = (V_\pi, E_\pi)$, where

$$V_\pi = \{v \in V : v.\pi \neq NIL\} \cup \{s\}$$

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G_π is a **breadth-first tree** if V_π consists of the vertices reachable from s and for all $v \in V_\pi$, G_π contains a unique simple path from s to v that is also a shortest path from s to v in G.

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Lemma

After the execution of BFS, the predecessor graph G_π is a breadth-first tree.