6331 - Algorithms, Spring 2014, CSE, OSU
Lecture 5: Red-black trees

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Red-black trees

For every node x:

- $x.color$: red or black
- $x.k$: key
- $x.p$: pointer to the parent of x
- $x.left$: pointer to the left child of x
- $x.right$: pointer to the right child of x

Properties of binary search trees

Let x be a node.

- For any node y in the left subtree of x, we have $y.key \leq x.key$.
- For any node y in the right subtree of x, we have $y.key \geq x.key$.

Properties of red-black trees

Properties:

1. Every node is either red or black.
2. The root is black.
3. Every leaf is black, and is represented by NIL.
4. If a node is red, then both its children are black.
5. For each node x, all paths from x to a descendant leaf of x contain the same number of black nodes.

Black-height

For a node x, the *black-height* of x, denoted $\mathrm{bh}(x)$ is the number of black nodes on any path from, but not including, x, to a descendant leaf of x.

The height of a red-black tree

Lemma

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Each child of a node x has black-height at least $\mathrm{bh}(x) - 1$.
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Let h be the height of the tree. Then, h is at most twice the black-height of the root.

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$n \geq 2^{h/2} - 1$.

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$n \geq 2^{h/2} - 1$.

$h = O(\log n)$. □

Implications

The operations Search, Minimum, Maximum, Successor, and Predecessor take time $O(\log n)$.

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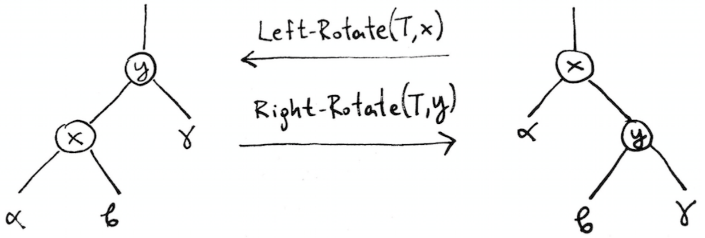
What about Insertion and Deletion?

Rotations

```
Left-Rotate(T, x)
    y = x.right
    x.right = y.left
    if y.left ≠ NIL
        y.left.p = x
    y.p = x.p
    if x.p = NIL
        T.root = y
    elseif x = x.p.left
        x.p.left = y
    else x.p.right = y
    y.left = x
    x.p = y
```

Rotations



Insertion

RB-Insert(T, z)

$y = NIL$
$x = T.root$
while $x \neq NIL$
    $y = x$
    if $z.key < x.key$
        $x = x.left$
    else $x = x.right$
$z.p = y$
if $y = NIL$
    $T.root = z$
elseif $z.key < y.key$
    $y.left = z$
else $y.right = z$
$z.left = NIL$
$z.right = NIL$
$z.color = RED$
RB-Insert-Fixup(T, z)

Insertion

Does RB-Insert create a valid Red-black tree?

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What can go wrong?

Insertion

Does RB-Insert create a valid Red-black tree?

What can go wrong?

- If the parent of z is RED, then we have two consecutive RED nodes.
- If T is empty, then after the insertion the root is RED.

Fixup

```
RB-Insert-Fixup(T, z)
  while z.p.color = RED
    if z.p = z.p.p.left
      y = z.p.p.right
      if y.color = RED
        z.p.color = BLACK
        y.color = BLACK
        z.p.p.color = RED
        z = z.p.p
      else
        if z = z.p.right
          z = z.p
          Left-Rotate(T.z)
        z.p.color = BLACK
        z.p.p.color = RED
        Right-Rotate(T, z.p.p)
    else (same with "left" and "right" exchanged)
  T.root.color = BLACK
```

Invariants of Fixup

(a) Node z is red.

(b) If $z.p$ is the root, then $z.p$ is black.

(c) If the tree violates any of the properties, then it violates at most one of them, and the violation is either 2, or 4.

- If it violates property 2, then z is the root and is red.
- If it violates property 4, then z and $z.p$ are red, and no other node violates property 4.

Invariants of Fixup

Initialization?

(a) Node z is red.

(b) If $z.p$ is the root, then $z.p$ is black.

(c) If the tree violates any of the properties, then it violates at most one of them, and the violation is either 2, or 4.

- If it violates property 2, then z is the root and is red.
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Invariants of Fixup

Termination?

(a) Node z is red.

(b) If $z.p$ is the root, then $z.p$ is black.

(c) If the tree violates any of the properties, then it violates at most one of them, and the violation is either 2, or 4.

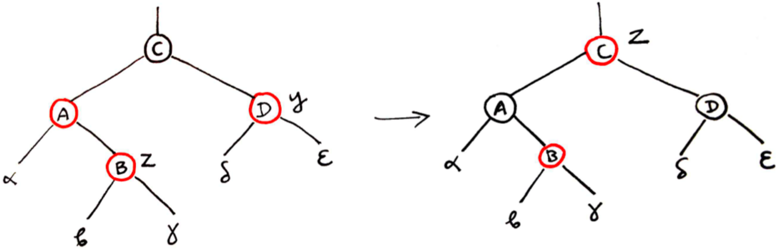
- If it violates property 2, then z is the root and is red.
- If it violates property 4, then z and $z.p$ are red, and no other node violates property 4.

Maintenance of invariants of Fixup

Assume w.l.o.g. that $z.p$ is a left child.
The other case is symmetric.

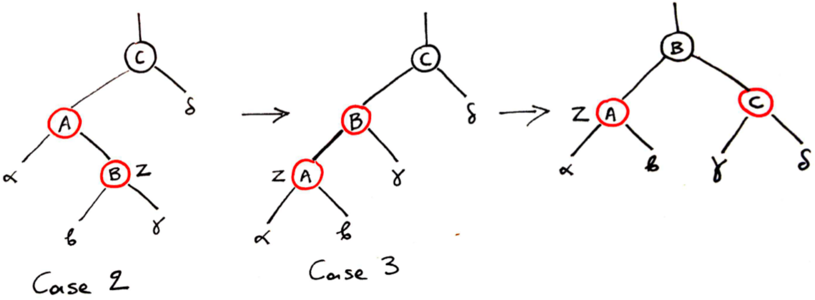
Maintenance of invariants of Fixup

Case 1: z's uncle y is red.



Maintenance of invariants of Fixup

Case 2: z's uncle y is black and z is a right child.
Case 3: z's uncle y is black and z is a left child.



Running time

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The running time of RB-Insert-Fixup is $O(\log n)$.

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Therefore, the running time of RB-Insert is $O(\log n)$.