6332 - Advanced algorithms, Spring 2015, CSE, OSU
Homework 3
Instructor: Anastasios Sidiropoulos
Due date: March 25, 2015

Problem 1 (TSP). An input to the Traveling Salesperson Problem (TSP) consists of a connected undirected n-vertex graph $G = (V, E)$ together with a non-negative function $w : E \to \mathbb{R}_{\geq 0}$, where for any $\{u, v\} \in E$, $w(\{u, v\})$ is the *length* of the edge $\{u, v\}$. For any $u, v \in V$ let $d_G(u, v)$ denote the shortest-path distance between u and v, i.e. the minimum length of a path between u and v in G. The goal is to find a *tour* of minimum length that visits all vertices and returns to the starting vertex. Formally, the goal is to find a bijection $\sigma : V \to \{1, \ldots, n\}$ minimizing the following objective function:

$$d_G(\sigma(n), \sigma(1)) + \sum_{i=1}^{n-1} d_G(\sigma(i), \sigma(i+1)).$$

(a) A Minimum Spanning Tree (MST) in G is a tree on minimum total length that spans all vertices. Show that an optimum solution for TSP has cost at most twice the the total length of a MST.

(b) Use part (a) to obtain a polynomial-time 2-approximation algorithm for TSP. Note that a MST in G can be computed in polynomial time.

Problem 2. (Vertex Cover) Let $G = (V, E)$ be an instance to the Vertex Cover problem, with $V = \{1, \ldots, n\}$. Recall that the natural linear programming relaxation for the Vertex Cover is the following:

$$\begin{array}{rll} \text{minimize} & \sum_{i \in V} x_i & \\ \text{subject to} & x_i + x_j \geq 1 & \text{for all } \{i, j\} \in E \\ & x_i \in [0, 1] & \text{for all } i \in V \end{array}$$

(a) Let G be the complete graph on n vertices. Show that the above LP relaxation has a solution with total cost $n/2$. Conclude that the integrality gap of the LP relaxation is at least $2 - o(1)$.

(b) Let $k \geq 2$. A k-uniform hypergraph is a pair $G = (V, E)$ where $V = \{1, \ldots, n\}$ is a finite set and $E \subseteq \binom{V}{k}$, that is E is a set of subsets of V each containing exactly k vertices. Note that a 2-uniform hypergraph is simply a graph. The Vertex Cover problem on k-uniform hypergraphs can be defined analogously to the graph case: The input consists of a k-uniform hypergraph and the goal is to find a minimum subset S of vertices such that every element in E intersects S. Derive a polynomial-time k-approximation algorithm for Vertex Cover on k-uniform hypergraphs.

(c) Give a linear programming relaxation for Vertex Cover on k-uniform hypergraphs. Show that the integrality gap of your relaxation is at most k and at least $k - o(1)$.

Problem 3. (Max-Cut with general weights) We saw in class a 2-approximation algorithm for Max-Cut on graphs with non-negative weights. The algorithm simply picks a random bi-partition of the vertex set. We showed that the expected weight of the resulting cut is half of the total weight of all edges in the graph. Suppose that the graph is allowed to have edges with both positive and negative weights. Is this still a 2-approximation?