# 6331 - Algorithms, CSE, OSU

## Heapsort

Instructor: Anastasios Sidiropoulos

# Sorting

Given an array of integers $A[1 \ldots n]$, rearrange its elements so that

$$A[1] \leq A[2] \leq \ldots \leq A[n].$$

# A simple sorting algorithm

Bubble-Sort

```
repeat
  swapped = false
  for i = 1 to n − 1 do
    if A[i] > A[i + 1] then
      swap(A[i], A[i + 1])
      swapped = true
    end if
  end for
until not swapped
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What is the worst-case time complexity of this algorithm?

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What is the worst-case time complexity of this algorithm?

We can do much better!

# Heaps

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- A heap is stored in an array $A[1 \ldots n]$.
- The root is $A[1]$.
- $\text{parent}(i) = i/2$.
- $\text{left-child}(i) = 2i$.
- $\text{right-child}(i) = 2i + 1$.

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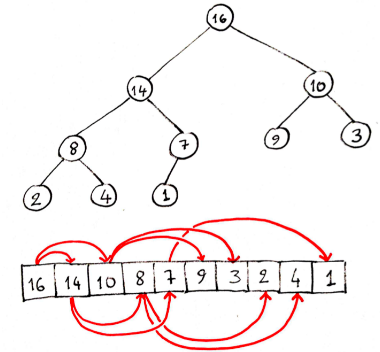

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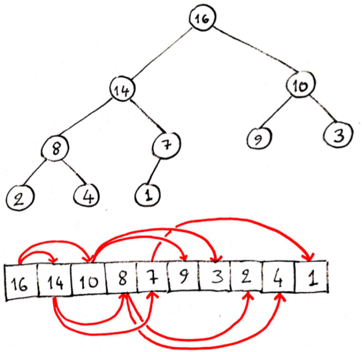


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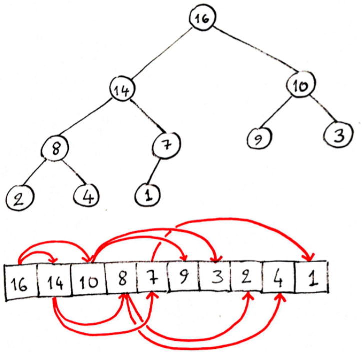


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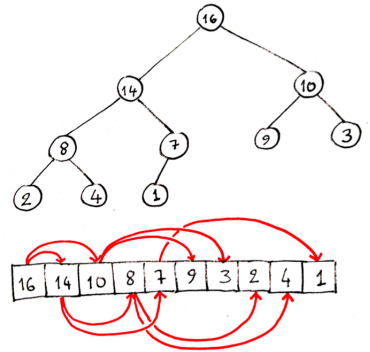


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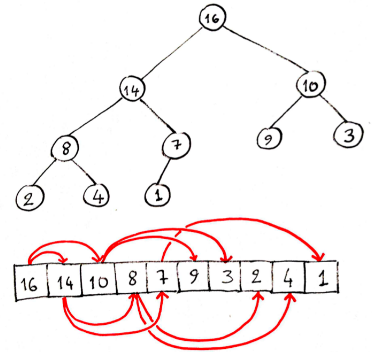


- Where is the maximum element in the tree?
- Where is the maximum element in the array?
- Where is the minimum element in the tree?
- Where is are the leaves in the array?

# Height

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What is the height of a heap?

# Building and using heaps

- Procedure Max-Heapify (auxiliary procedure)
- Procedure Build-Max-Heap (building a max-heap)
- Procedure Heap-Sort (sorting using a heap)

# Maintaining the max-heap property

Suppose that the subtrees rooted at left-child($i$) and right-child($i$) are max-heaps.

However, $i$ might violate the max-heap property.
E.g., $A[i] < A[\text{left-child}(i)]$.

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How can we enforce the max-heap property?

# Maintaining the max-heap property

```
Procedure Max-Heapify(A, i)
  l = left-child(i)
  r = right-child(i)
  if l ≤ n and A[l] > A[i]
    largest = l
  else largest = i
  if r ≤ n and A[r] > largest
    largest = r
  if largest ≠ i
    exchange A[i] with A[largest]
    Max-Heapify(A, largest)
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- Worst-case depth of recursion = height of $i$.
- Worst-case running time is $O(\text{height}(i))$.
- Worst-case running time is $O(\log(n))$.
- Is this tight?

# Building a heap

Procedure Build-Max-Heap($A$)
  for $i = \lfloor n/2 \rfloor$ downto 1
    Max-Heapify($A, i$)

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- **Maintenance:** By the loop invariant, the children of $i$ are roots of max-heaps. Therefore, running Max-Heapify makes $i$ the root of a max-heap.
- **Termination:** $i = 1$. By the loop invariant, 1 is the root of a heap.

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- This is not asymptotically tight!

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- A heap has height $\lfloor \log(n) \rfloor$.
- There are at most $\lceil n/2^{h+1} \rceil$ nodes of height $h$.
- Total running time:

$$\sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left( n \sum_{h=0}^{\infty} \frac{h}{2^h} \right) = O(n).$$

# Sorting using a heap

```
Procedure Heapsort(A)
  Build-Max-Heap(A)
  for i = A.length downto 2
    exchange A[1] with A[i]
    A.heap-size = A.heap-size − 1
    Max-Heapify(A, 1)
```

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- Each call to Max-Heapify takes time $O(\log n)$.
- Total running time $O(n\log(n))$.

# Priority queues

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There are max-priority queues, and min-priority queues.

Operations of a max-priority queue:

- Insert$(S, x)$: $S = S \cup \{x\}$.
- Maximum$(S)$: Return the element in $S$ with the maximum key.
- Extract-Max$(S)$: Removes and returns the element in $S$ with the maximum key.
- Increase-Key$(S, x, k)$: Increases the value of the key of $x$ to $k$, assuming that $k$ is larger than the current value.

# Implementing a max-priority queue using a max-heap

Procedure Heap-Maximum($A$)

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Procedure Heap-Maximum($A$)
  return $A[1]$

# Implementing a max-priority queue using a max-heap

Procedure Heap-Extract-Max($A$)

# Implementing a max-priority queue using a max-heap

Procedure Heap-Extract-Max($A$)
  if $n < 1$
    error "empty heap"
  max $= A[1]$
  $A[1] = A[n]$
  $n = n - 1$
  Max-Heapify($A, 1$)
  return max

# Implementing a max-priority queue using a max-heap

Procedure Heap-Increase-Key($A, i, \text{key}$)

# Implementing a max-priority queue using a max-heap

```
Procedure Heap-Increase-Key(A, i, key)
  if key < A[i]
    error
  A[i] = key
  while i > 1 and A[parent(i)] < A[i]
    exchange A[i] with A[parent(i)]
    i = parent(i)
```

# Implementing a max-priority queue using a max-heap

Procedure Max-Heap-Insert($A$, key)

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Procedure Max-Heap-Insert($A$, key)

$n = n + 1$

$A[n] = -\infty$

Heap-Increase-Key($A$, $n$, key)