

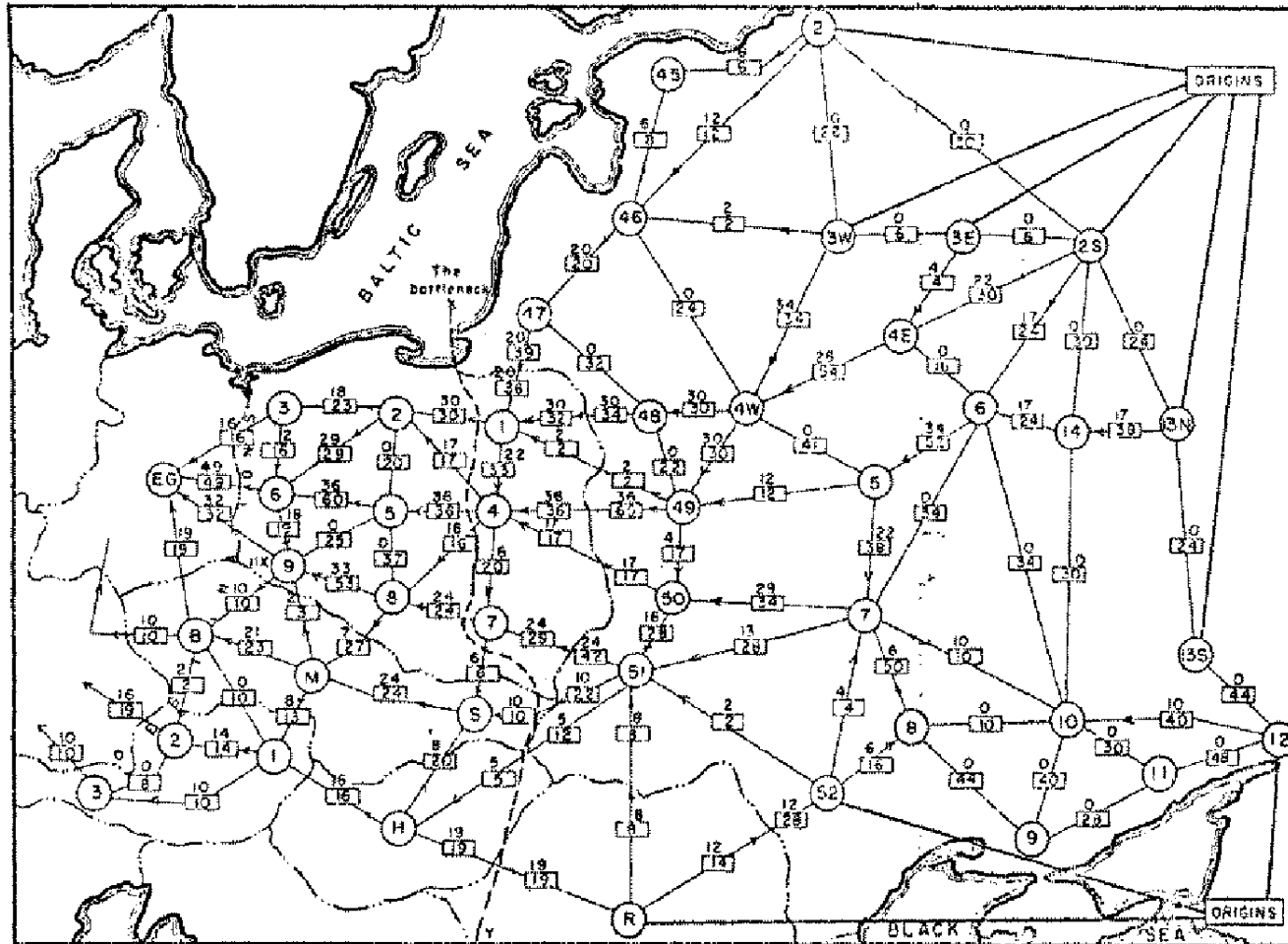
Chapter 7

Network Flow



Slides by Kevin Wayne.
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Soviet Rail Network, 1955



Reference: *On the history of the transportation and maximum flow problems.*
Alexander Schrijver in *Math Programming*, 91: 3, 2002.

Maximum Flow and Minimum Cut

Max flow and min cut.

- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

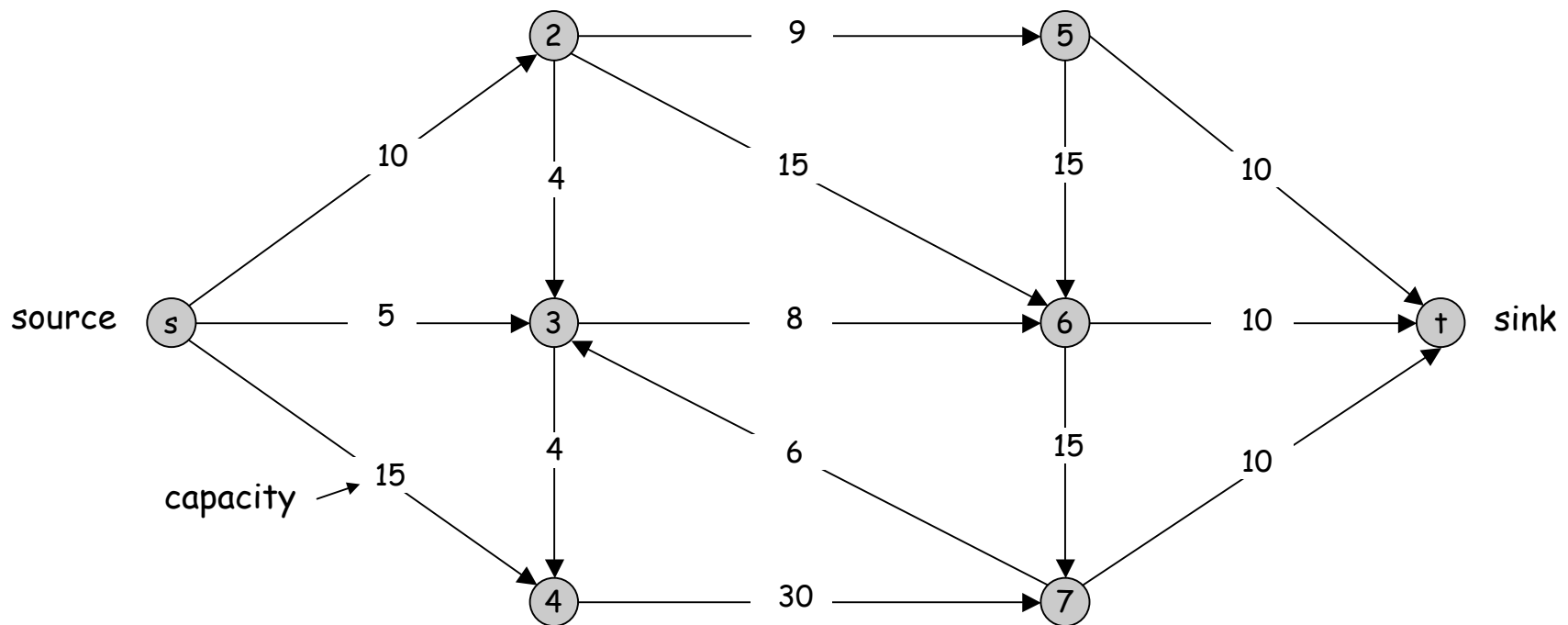
Nontrivial applications / reductions.

- Data mining.
- Open-pit mining.
- Project selection.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Network reliability.
- Distributed computing.
- Egalitarian stable matching.
- Security of statistical data.
- Network intrusion detection.
- Multi-camera scene reconstruction.
- Many many more ...

Minimum Cut Problem

Flow network.

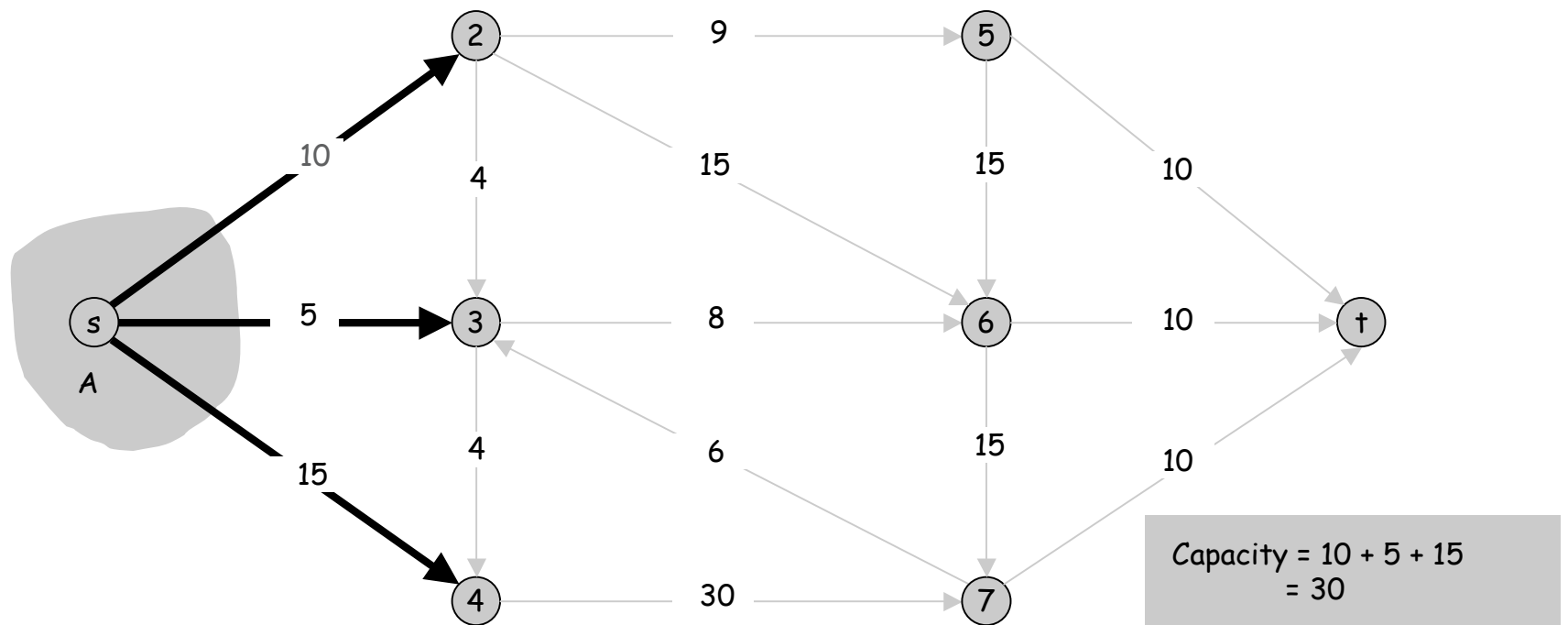
- Abstraction for material **flowing** through the edges.
- $G = (V, E)$ = directed graph, no parallel edges.
- Two distinguished nodes: s = source, t = sink.
- $c(e)$ = capacity of edge e .



Cuts

Def. An **s-t cut** is a partition (A, B) of V with $s \in A$ and $t \in B$.

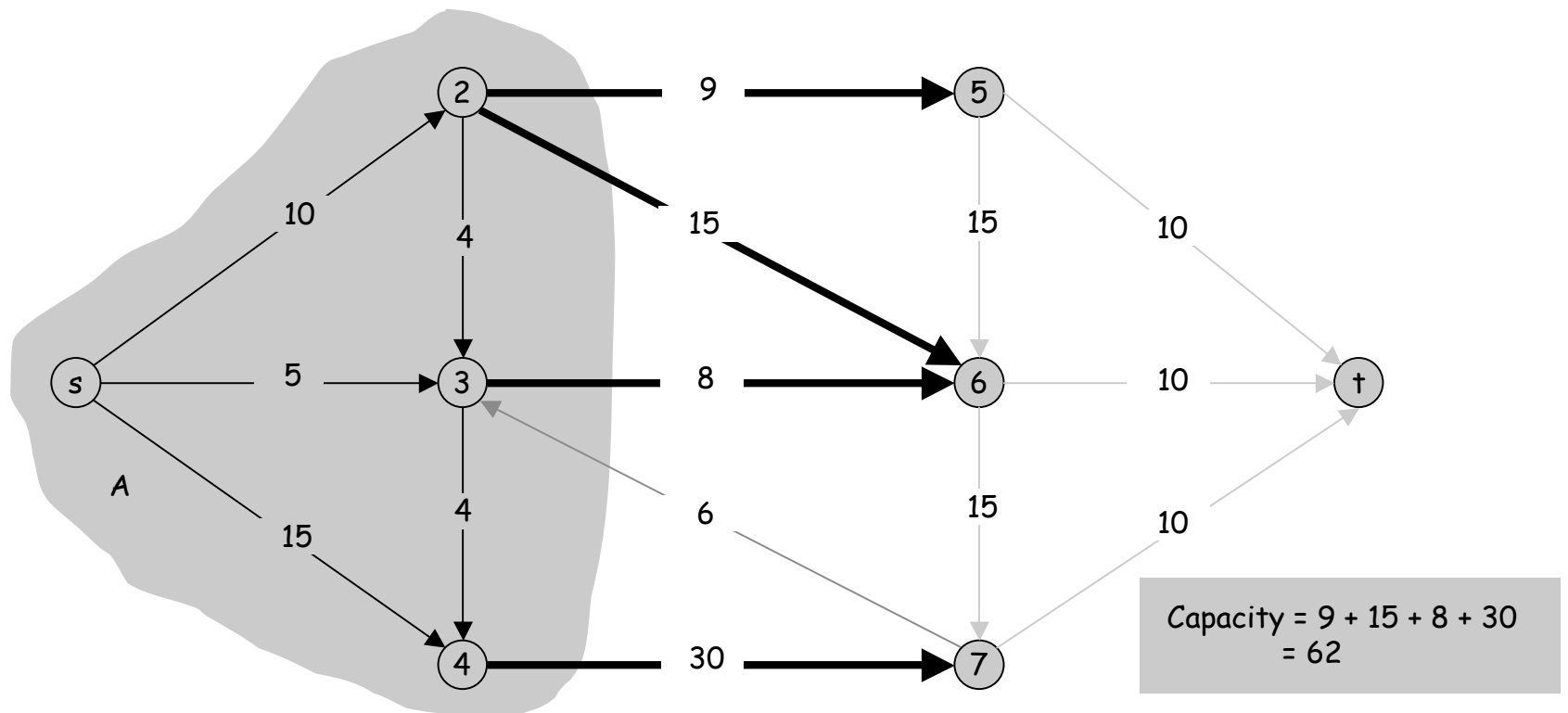
Def. The **capacity** of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



Cuts

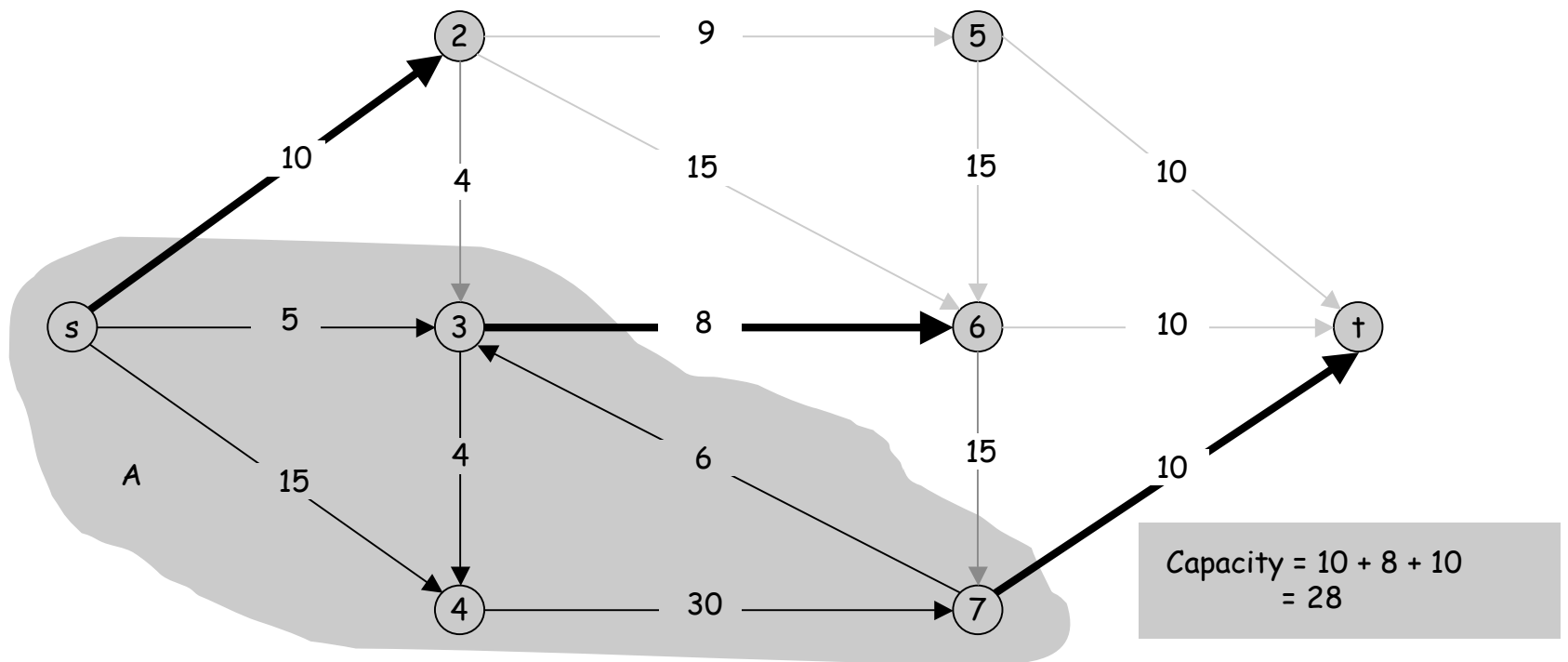
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Def. The **capacity** of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



Minimum Cut Problem

Min s-t cut problem. Find an s-t cut of minimum capacity.

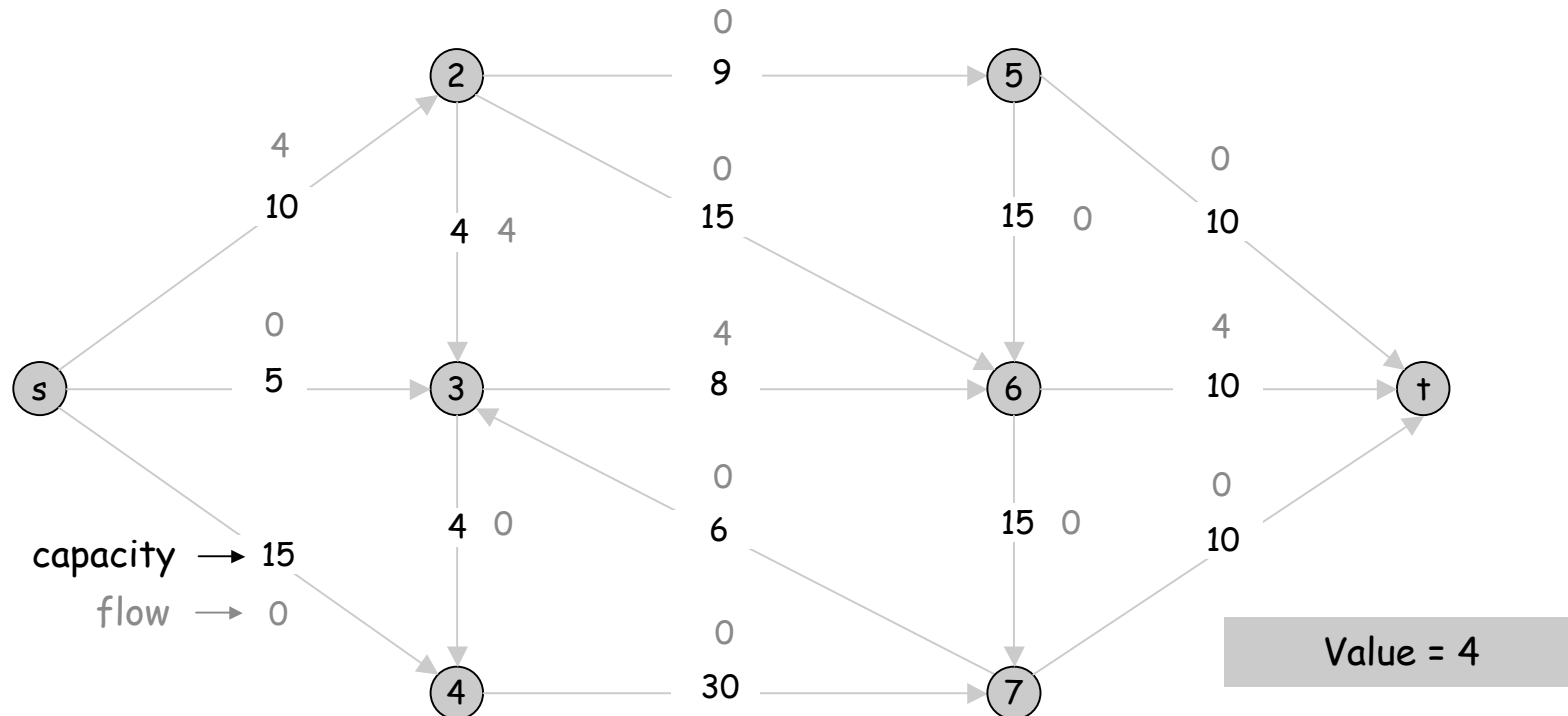


Flows

Def. An **s-t flow** is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ [capacity]
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [conservation]

Def. The **value** of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.

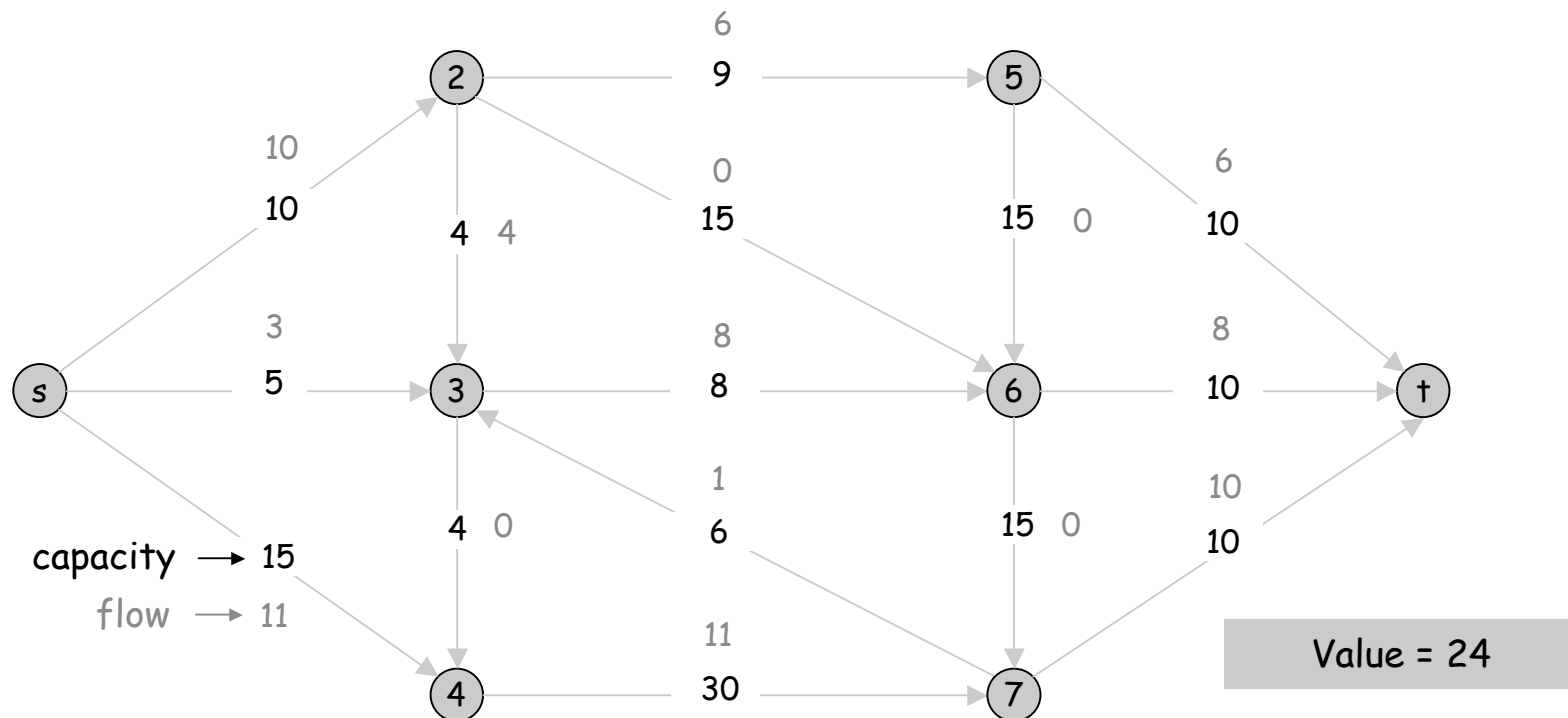


Flows

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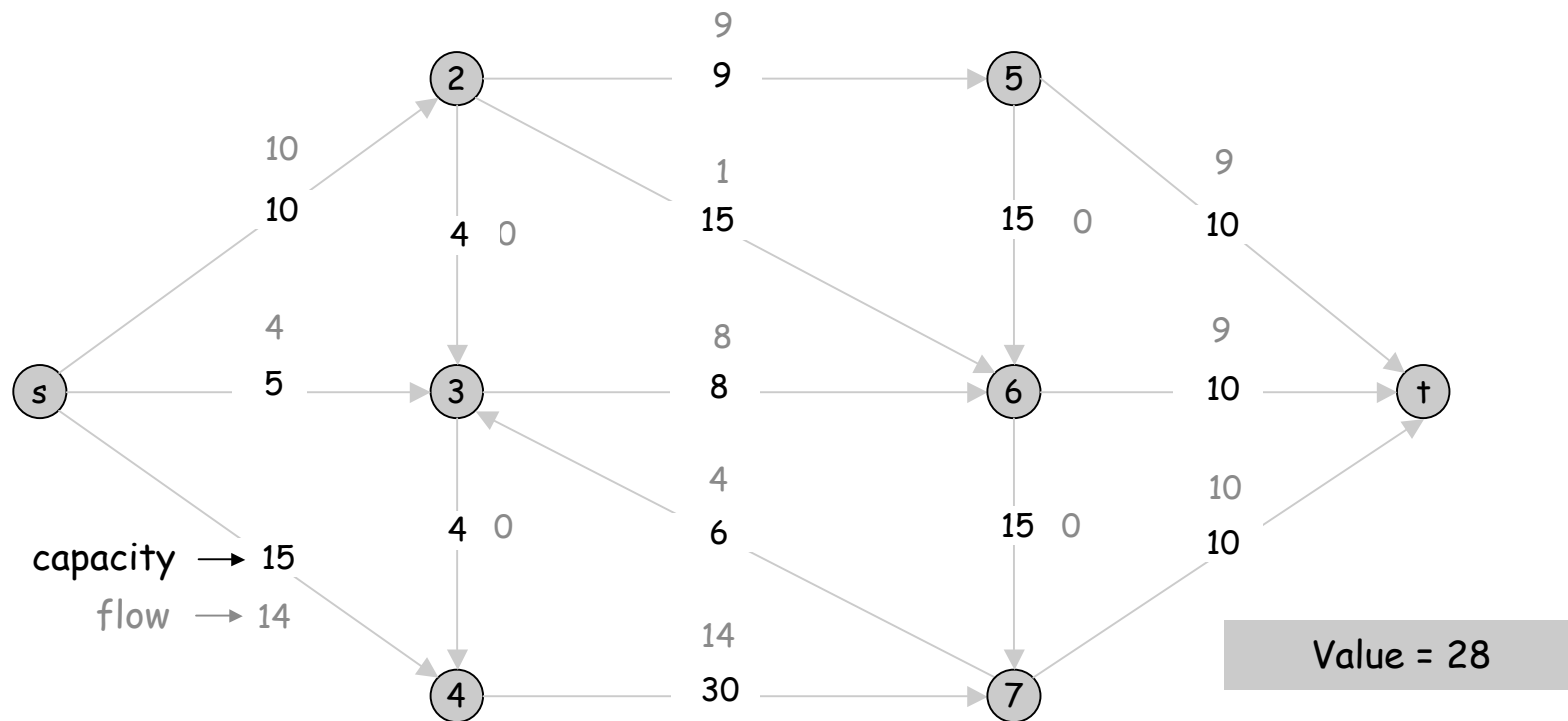
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Def. The **value** of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



Maximum Flow Problem

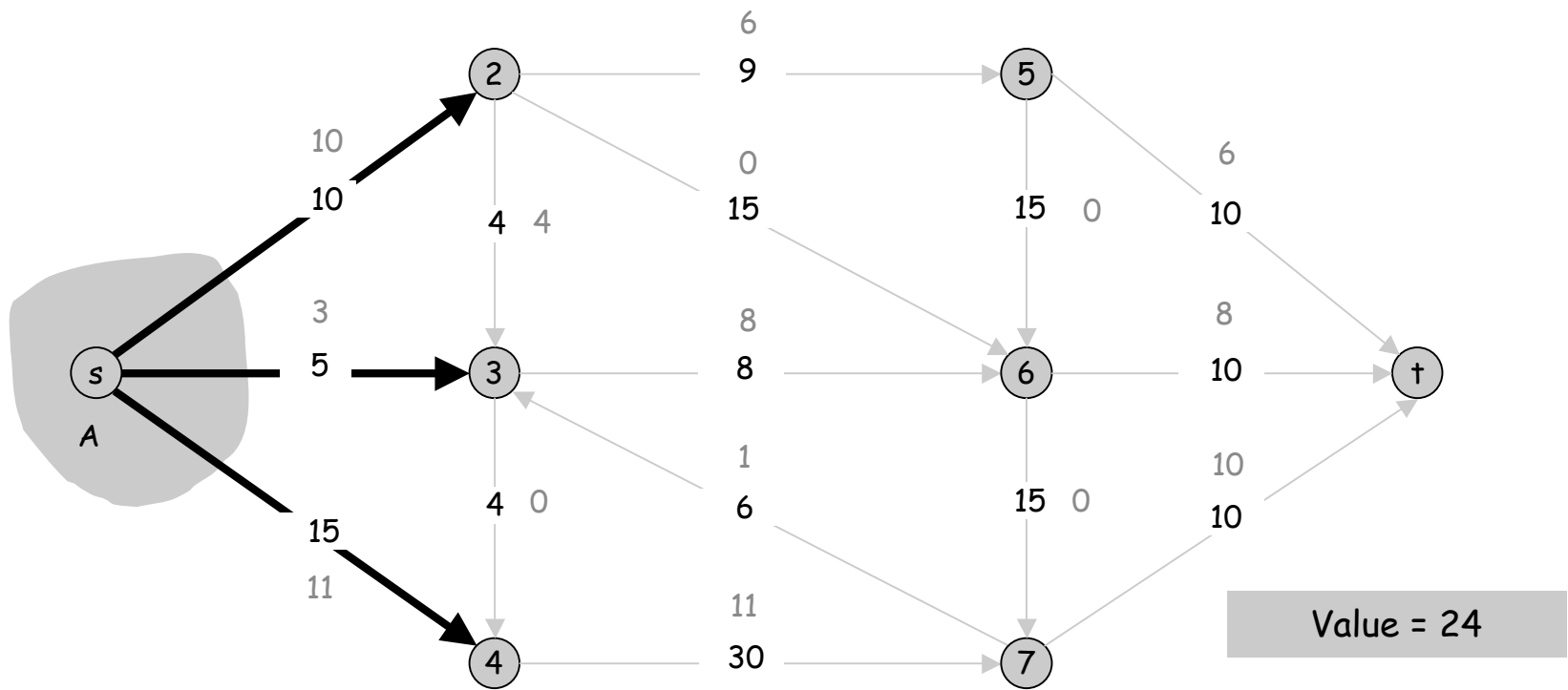
Max flow problem. Find s-t flow of maximum value.



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s - t cut. Then, the net flow sent across the cut is equal to the amount leaving s .

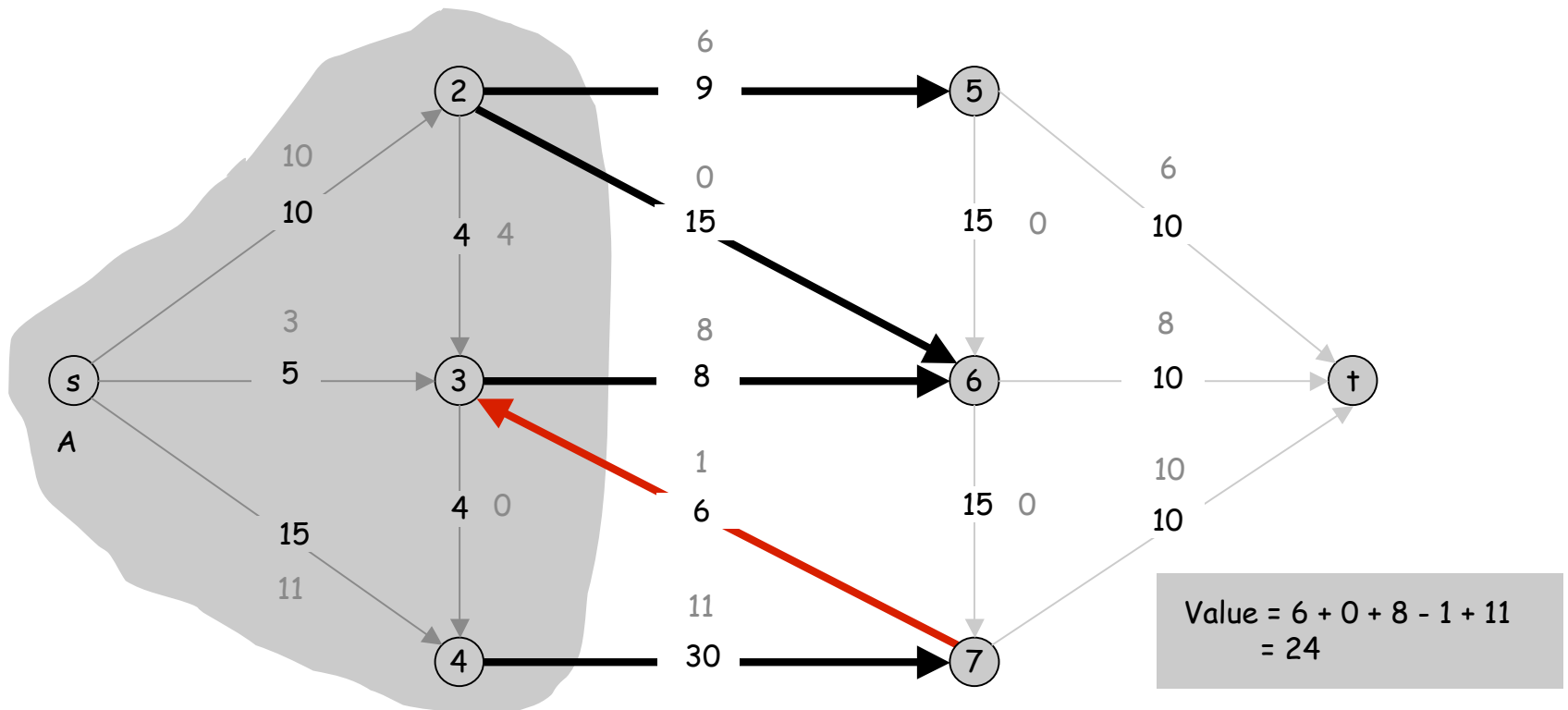
$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



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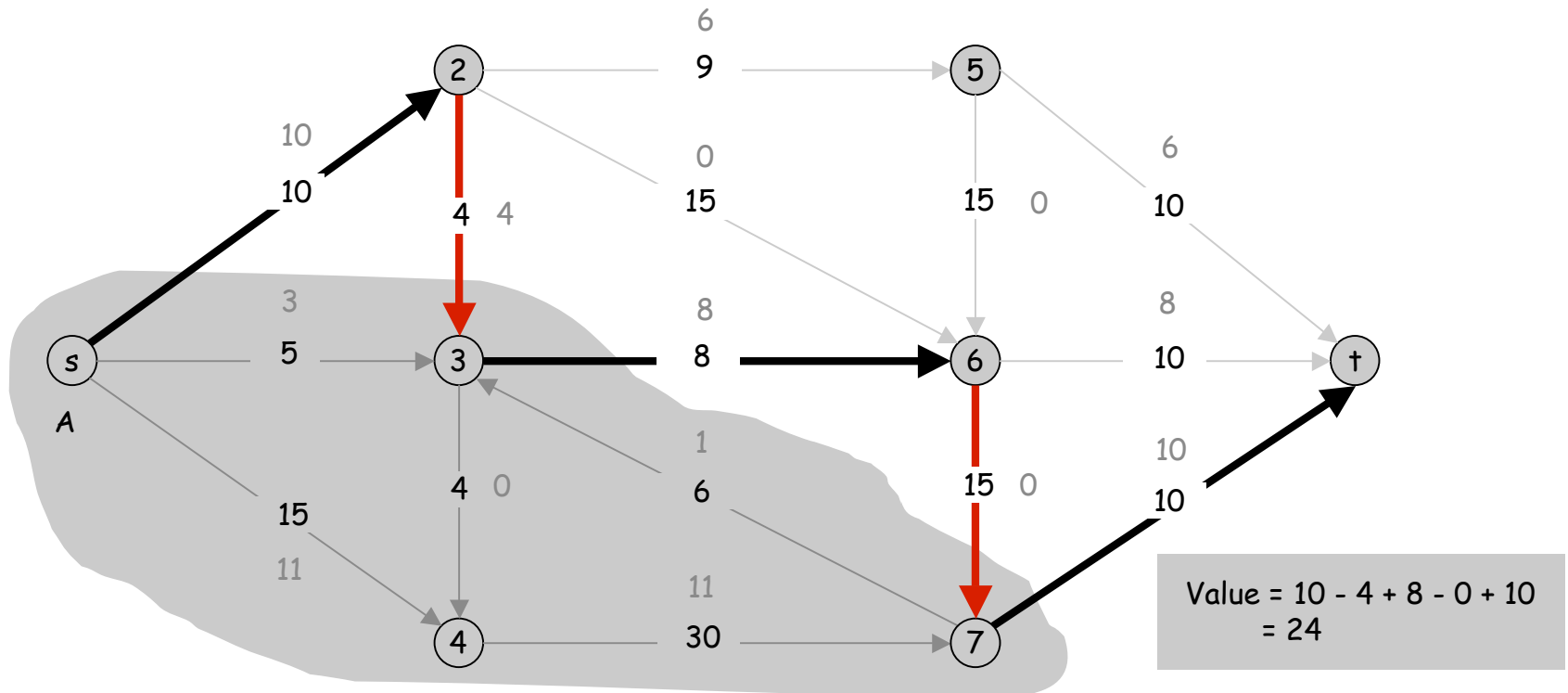
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Flows and Cuts

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Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s - t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

by flow conservation, all terms
except $v = s$ are 0 \rightarrow

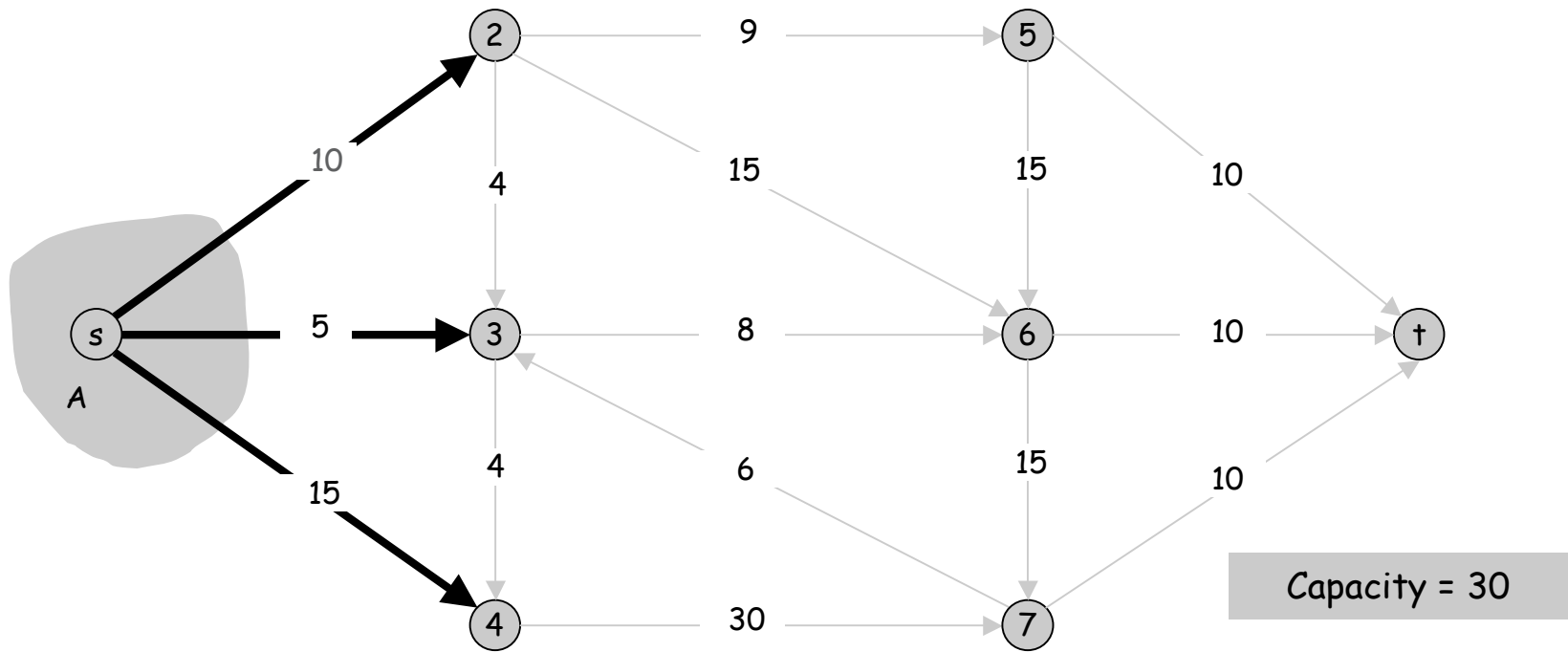
$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e).$$

Flows and Cuts

Weak duality. Let f be any flow, and let (A, B) be any s - t cut. Then the value of the flow is at most the capacity of the cut.

Cut capacity = 30 \Rightarrow Flow value \leq 30

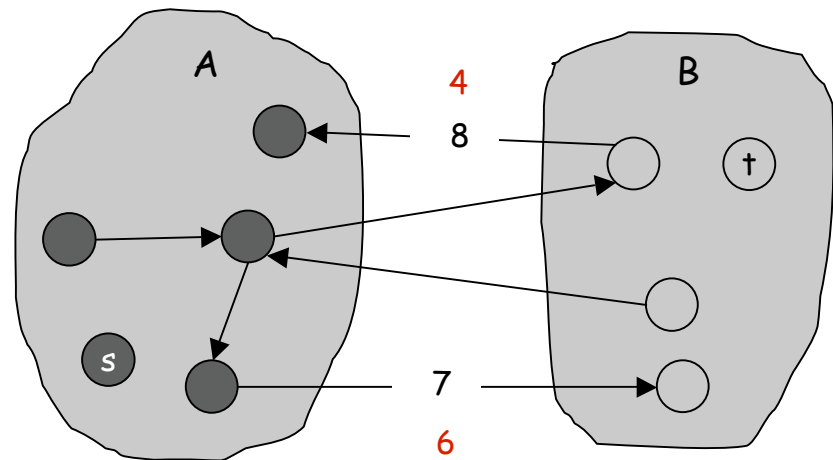


Flows and Cuts

Weak duality. Let f be any flow. Then, for any s - t cut (A, B) we have $v(f) \leq \text{cap}(A, B)$.

Pf.

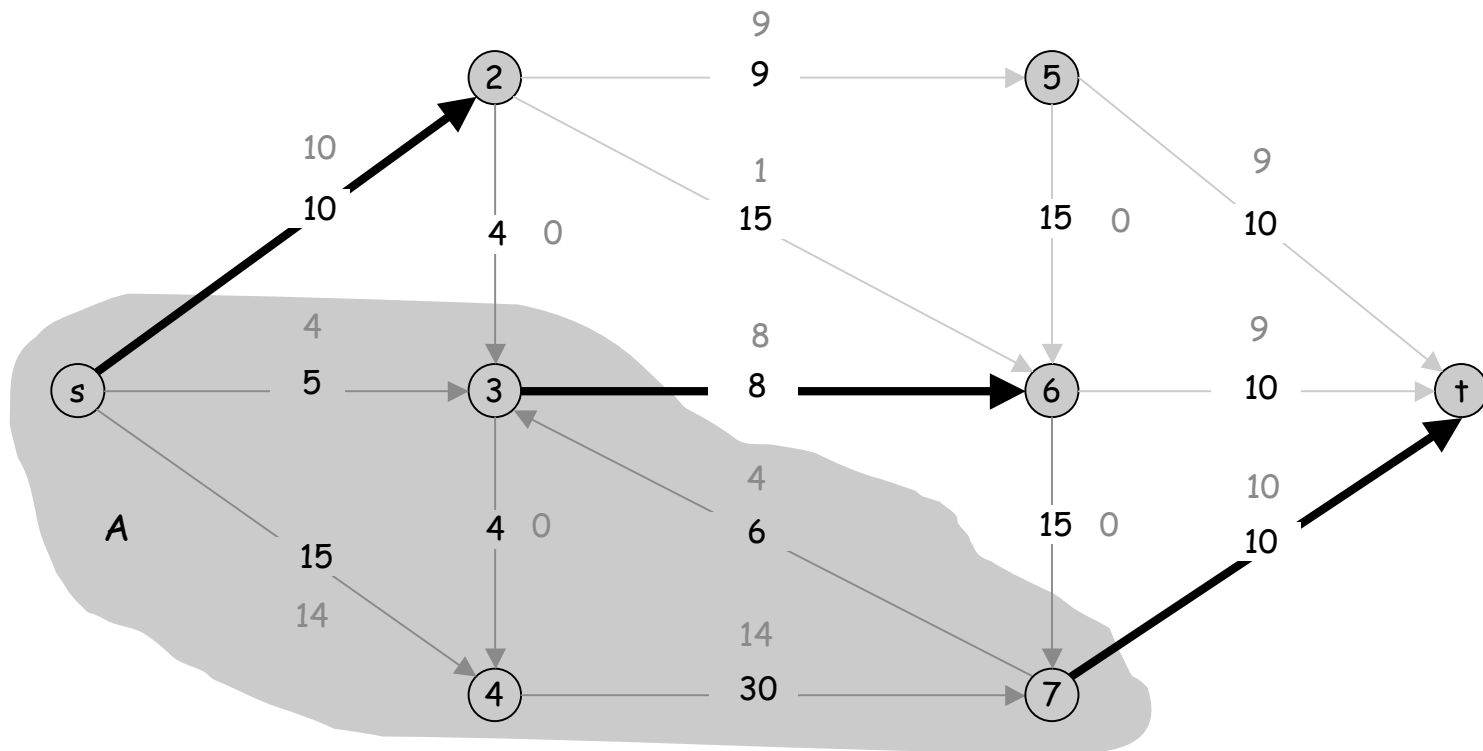
$$\begin{aligned}
 v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
 &\leq \sum_{e \text{ out of } A} c(e) \\
 &\leq \sum_{e \text{ out of } A} c(e) \\
 &= \text{cap}(A, B) \quad \blacksquare
 \end{aligned}$$



Certificate of Optimality

Corollary. Let f be any flow, and let (A, B) be any cut. If $v(f) = \text{cap}(A, B)$, then f is a max flow and (A, B) is a min cut.

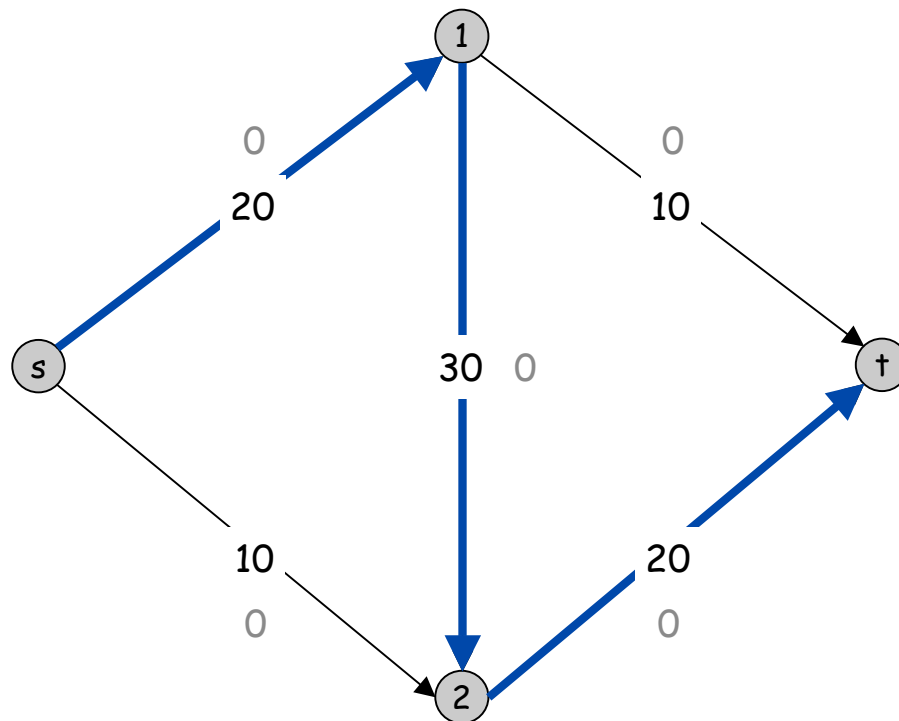
Value of flow = 28
 Cut capacity = 28 \Rightarrow Flow value \leq 28



Towards a Max Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for all edge $e \in E$.
- Find an s - t path P where each edge has $f(e) < c(e)$.
- Augment flow along path P .
- Repeat until you get stuck.

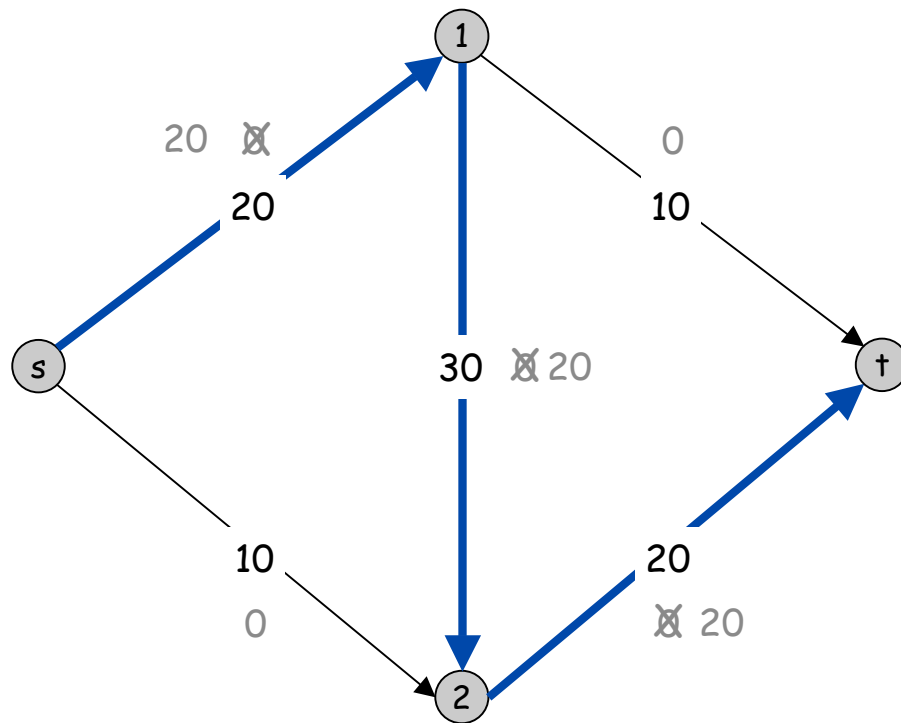


Flow value = 0

Towards a Max Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for all edge $e \in E$.
- Find an s - t path P where each edge has $f(e) < c(e)$.
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- Repeat until you get stuck.



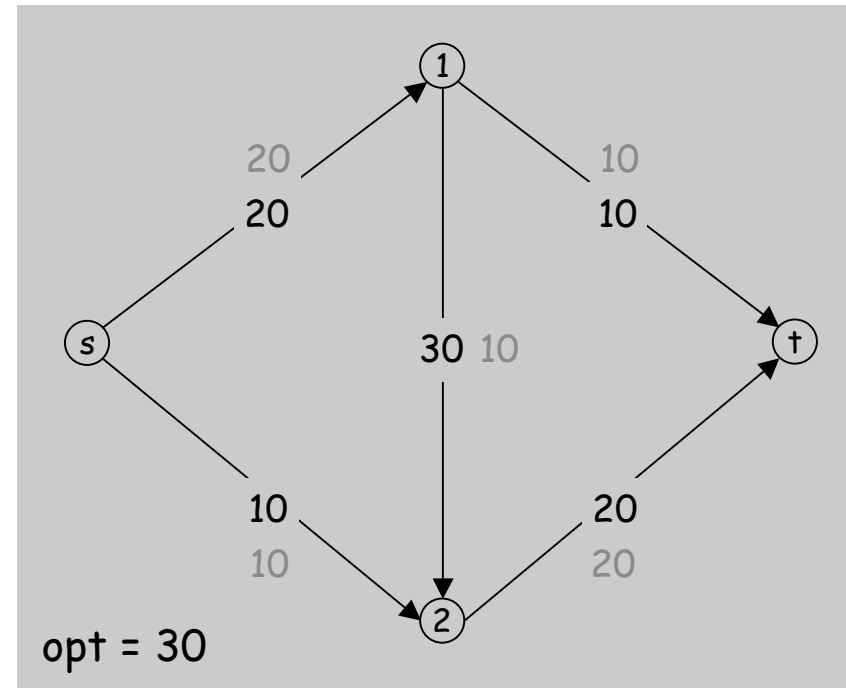
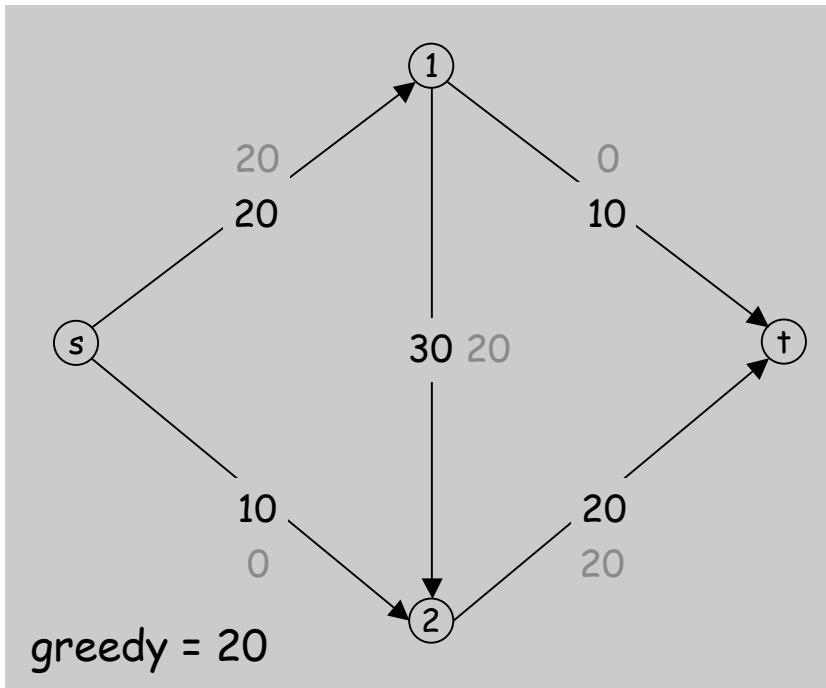
Flow value = 20

Towards a Max Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for all edge $e \in E$.
- Find an s - t path P where each edge has $f(e) < c(e)$.
- Augment flow along path P .
- Repeat until you get **stuck**.

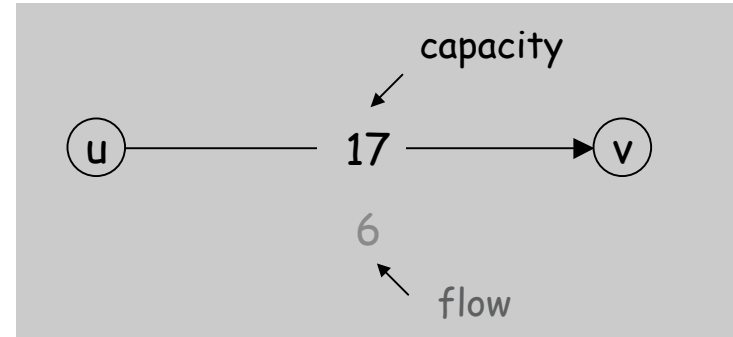
↖ locally optimality $\not\Rightarrow$ global optimality



Residual Graph

Original edge: $e = (u, v) \in E$.

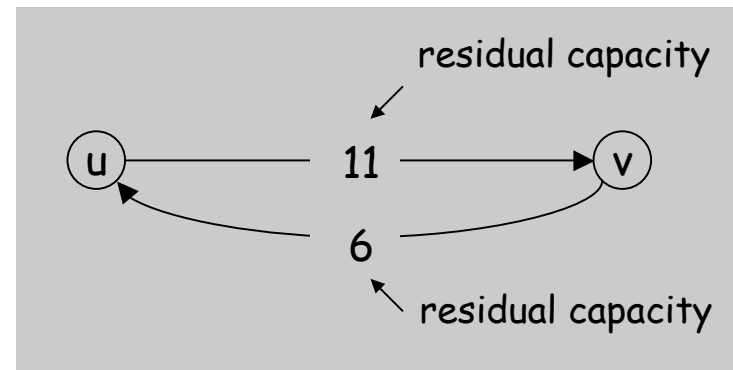
- Flow $f(e)$, capacity $c(e)$.



Residual edge.

- "Undo" flow sent.
- $e = (u, v)$ and $e^R = (v, u)$.
- Residual capacity:

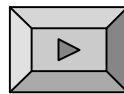
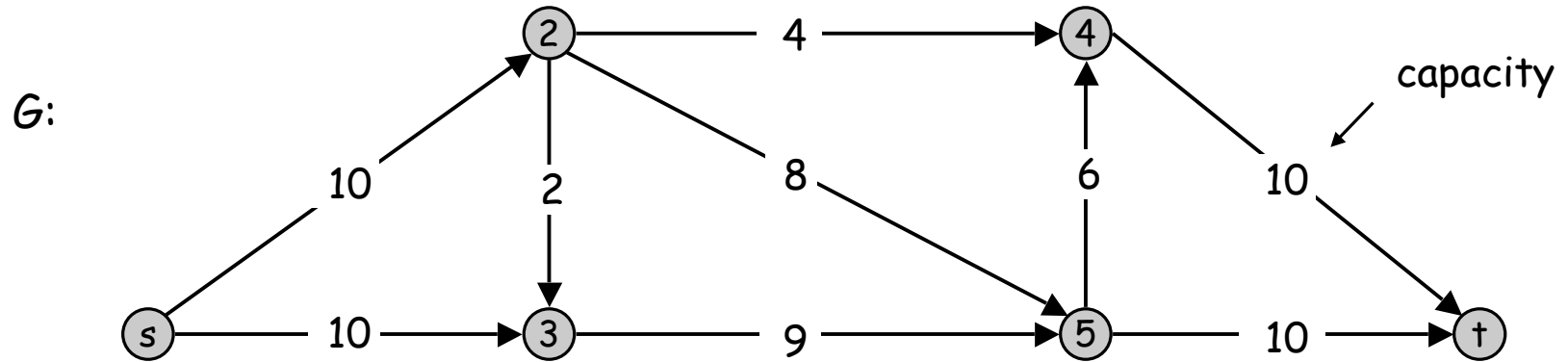
$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^R \in E \end{cases}$$



Residual graph: $G_f = (V, E_f)$.

- Residual edges with positive residual capacity.
- $E_f = \{e : f(e) < c(e)\} \cup \{e^R : f(e) > 0\}$.

Ford-Fulkerson Algorithm



Augmenting Path Algorithm

```
Augment(f, c, P) {  
  b ← bottleneck(P)  
  foreach e ∈ P {  
    if (e ∈ E) f(e) ← f(e) + b  
    else      f(eR) ← f(eR) - b  
  }  
  return f  
}
```

forward edge

reverse edge

```
Ford-Fulkerson(G, s, t, c) {  
  foreach e ∈ E f(e) ← 0  
  Gf ← residual graph  
  
  while (there exists augmenting path P) {  
    f ← Augment(f, c, P)  
    update Gf  
  }  
  return f  
}
```

Max-Flow Min-Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Elias-Feinstein-Shannon 1956, Ford-Fulkerson 1956]
The value of the max flow is equal to the value of the min cut.

Pf. We prove both simultaneously by showing TFAE:

- (i) There exists a cut (A, B) such that $v(f) = \text{cap}(A, B)$.
- (ii) Flow f is a max flow.
- (iii) There is no augmenting path relative to f .

(i) \Rightarrow (ii) This was the corollary to weak duality lemma.

(ii) \Rightarrow (iii) We show contrapositive.

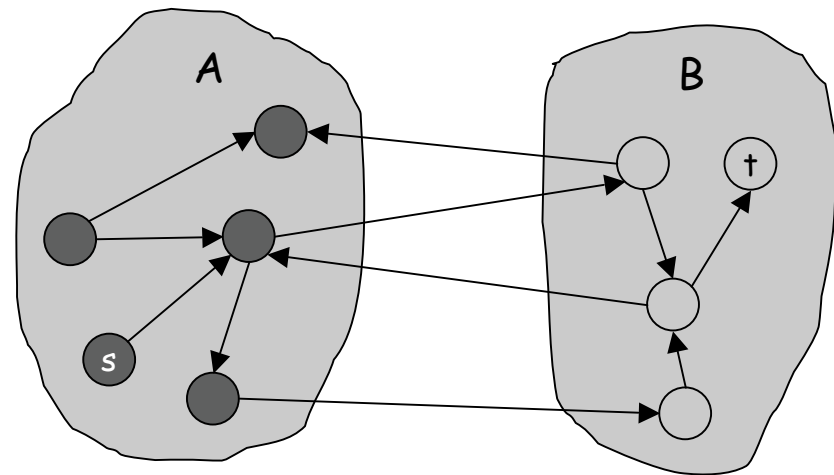
- Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.

Proof of Max-Flow Min-Cut Theorem

(iii) \Rightarrow (i)

- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s in residual graph.
- By definition of A , $s \in A$.
- By definition of f , $t \notin A$.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &= \sum_{e \text{ out of } A} c(e) \\ &= \text{cap}(A, B) \quad \blacksquare \end{aligned}$$



original network

Running Time

Assumption. All capacities are integers between 1 and C .

Invariant. Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \leq nC$ iterations.

Pf. Each augmentation increase value by at least 1. ■

Corollary. If $C = 1$, Ford-Fulkerson runs in $O(mn)$ time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value $f(e)$ is an integer.

Pf. Since algorithm terminates, theorem follows from invariant. ■

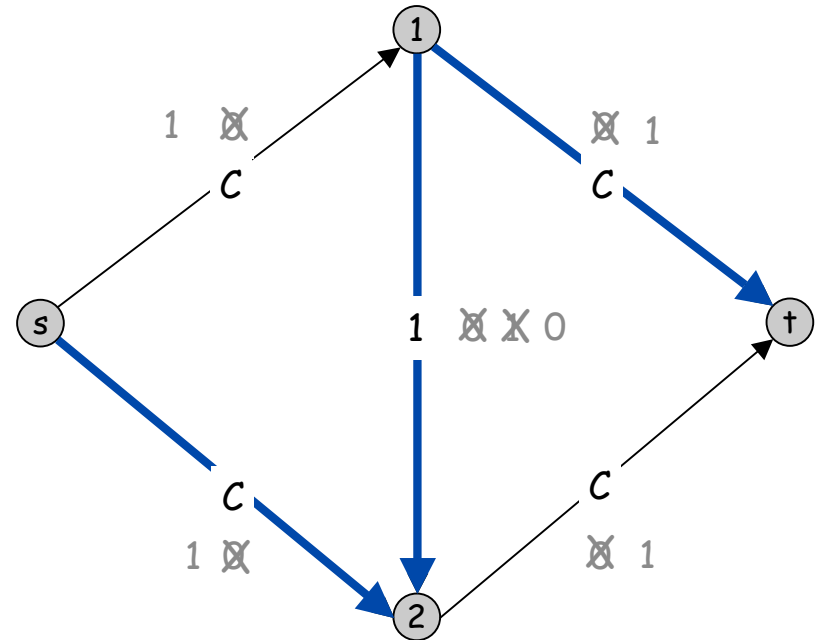
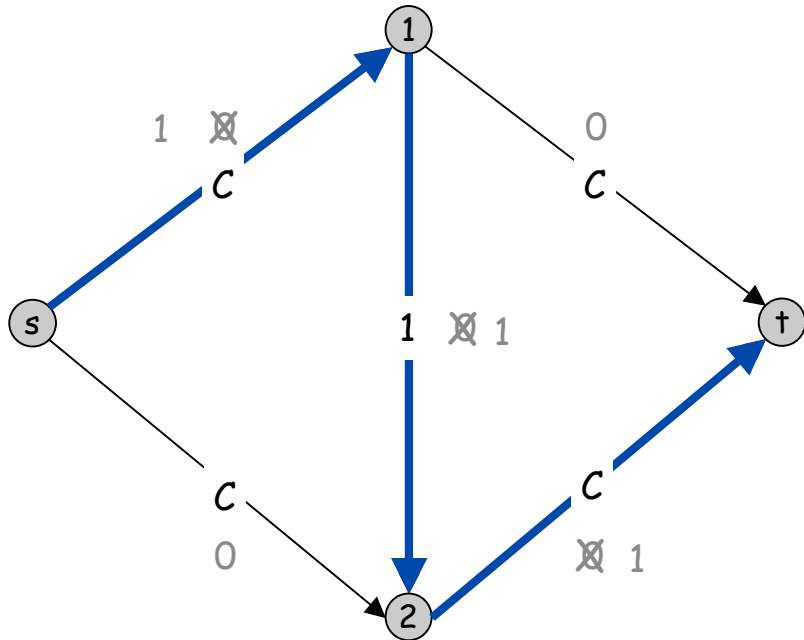
7.3 Choosing Good Augmenting Paths

Ford-Fulkerson: Exponential Number of Augmentations

Q. Is generic Ford-Fulkerson algorithm polynomial in input size?

$m, n,$ and $\log C$ \nearrow

A. No. If max capacity is C , then algorithm can take C iterations.



Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- If capacities are irrational, algorithm not guaranteed to terminate!

Goal: choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

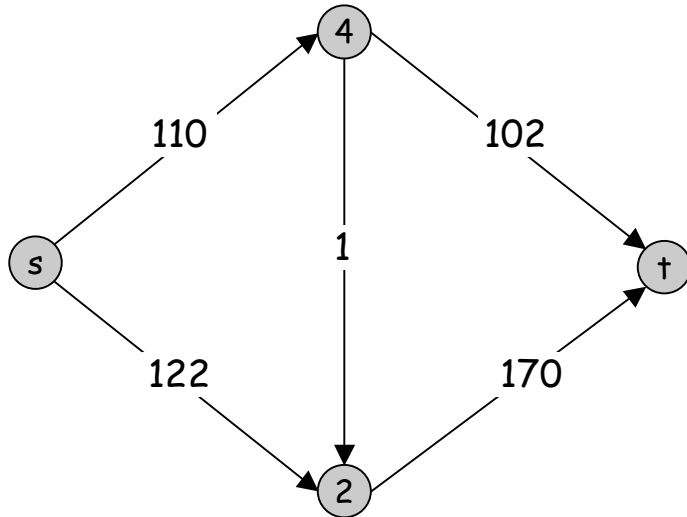
Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]

- Max bottleneck capacity.
- Sufficiently large bottleneck capacity.
- Fewest number of edges.

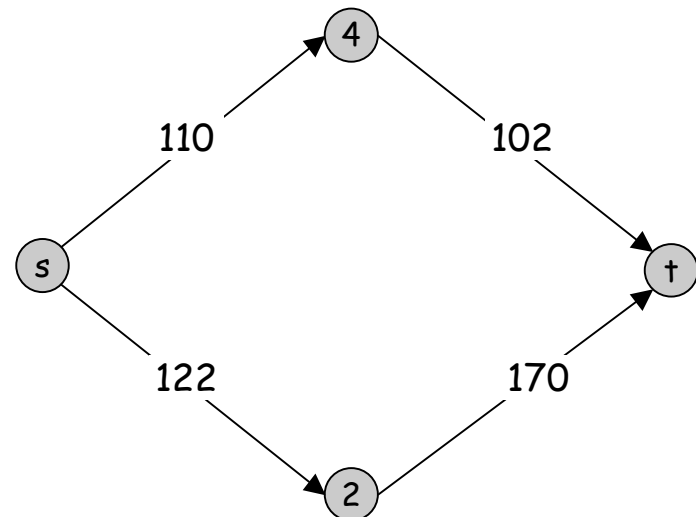
Capacity Scaling

Intuition. Choosing path with highest bottleneck capacity increases flow by max possible amount.

- Don't worry about finding exact highest bottleneck path.
- Maintain scaling parameter Δ .
- Let $G_f(\Delta)$ be the subgraph of the residual graph consisting of only arcs with capacity at least Δ .



G_f



$G_f(100)$

Capacity Scaling

```
Scaling-Max-Flow(G, s, t, c) {  
  foreach e ∈ E  f(e) ← 0  
  Δ ← smallest power of 2 greater than or equal to C  
  Gf ← residual graph  
  
  while (Δ ≥ 1) {  
    Gf(Δ) ← Δ-residual graph  
    while (there exists augmenting path P in Gf(Δ)) {  
      f ← augment(f, c, P)  
      update Gf(Δ)  
    }  
    Δ ← Δ / 2  
  }  
  return f  
}
```

Capacity Scaling: Correctness

Assumption. All edge capacities are integers between 1 and C .

Integrality invariant. All flow and residual capacity values are integral.

Correctness. If the algorithm terminates, then f is a max flow.

Pf.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths. ▪

Capacity Scaling: Running Time

Lemma 1. The outer while loop repeats $1 + \lceil \log_2 C \rceil$ times.

Pf. Initially $C \leq \Delta < 2C$. Δ decreases by a factor of 2 each iteration. ■

Lemma 2. Let f be the flow at the end of a Δ -scaling phase. Then the value of the maximum flow is at most $v(f) + m \Delta$. ← proof on next slide

Lemma 3. There are at most $2m$ augmentations per scaling phase.

- Let f be the flow at the end of the previous scaling phase.
- $L2 \Rightarrow v(f^*) \leq v(f) + m(2\Delta)$.
- Each augmentation in a Δ -phase increases $v(f)$ by at least Δ . ■

Theorem. The scaling max-flow algorithm finds a max flow in $O(m \log C)$ augmentations. It can be implemented to run in $O(m^2 \log C)$ time. ■

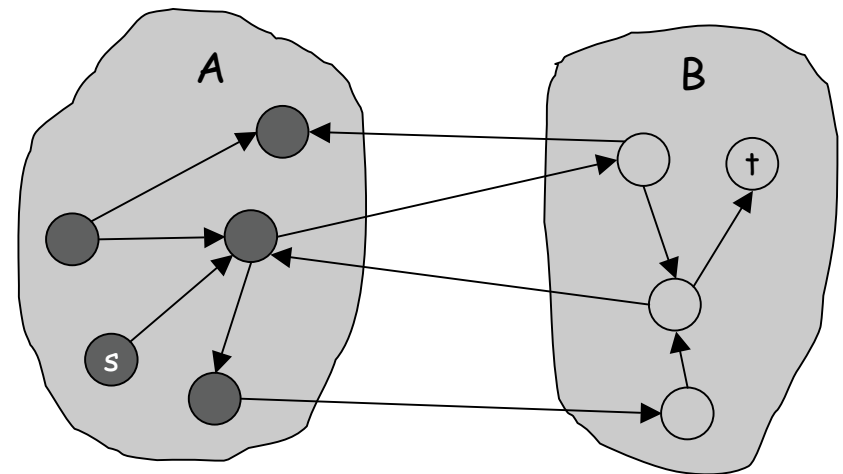
Capacity Scaling: Running Time

Lemma 2. Let f be the flow at the end of a Δ -scaling phase. Then value of the maximum flow is at most $v(f) + m \Delta$.

Pf. (almost identical to proof of max-flow min-cut theorem)

- We show that at the end of a Δ -phase, there exists a cut (A, B) such that $\text{cap}(A, B) \leq v(f) + m \Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of A , $s \in A$.
- By definition of f , $t \notin A$.

$$\begin{aligned}
 v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
 &\geq \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\
 &= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\
 &\geq \text{cap}(A, B) - m\Delta \quad \blacksquare
 \end{aligned}$$



original network