MATH 8500 Algorithmic Graph Theory, Spring 2017, OSU

Lecture 6: Maximum Bipartite Matching

Instructor: Anastasios Sidiropoulos

Scribe: Sherif ElAzzouni

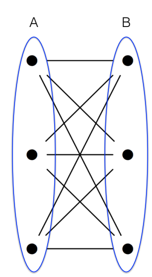
1 Problem Description

Input: $G(V, E)$.

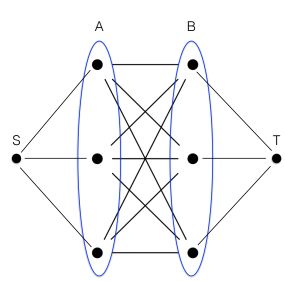
Goal: Find a matching $M \subseteq E$ maximizing $|M|$.

Definition: $M \subseteq E$ is a matching if no vertex is incident to two or more edges.

Bipartite Graph



One possible solution is to connect a source to partition A and a sink to partition B, Maximum matching can be found by solving the max-flow problem.



2 Definitions

- A vertex is "unmatched" or "exposed" w.r.t to matching M if v is not incident to any edges in M.
- A matching is "perfect" if no vertex is exposed.
- An "alternating path" w.r.t. some matching M is a path that alternates between edges in M and edges in $E \backslash M$.
- An "augmenting path" w.r.t. M is an alternating path in which the first and the last vertex are exposed.

Definition

Let P be an augmenting path w.r.t. M , "Augmenting path along P" means replacing M by

$$M' = M \triangle P = (M \backslash P) \cup (P \backslash M) \tag{1}$$

Lemma 2.1. *M' is a matching*

Proof. None of the edges incident to the path P are in the matching
When we augment M along P, the matching property will not be violated. □

Lemma 2.2. $|M'| = |M| + 1$

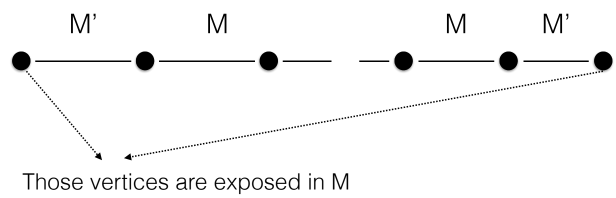
Theorem 2.3. *A matching M is maximum if and only if there is no augmenting path w.r.t. M*

Proof. "$\Rightarrow$"
If there is an augmenting path then $M' = P \triangle M$ is a bigger matching by the previous Lemmas, thus M is not maximum.
" $\Leftarrow$ "
(We will prove that if M is not maximum, then $\exists$ an augmenting path).
Let M' be a maximum matching such that $|M'| > |M|$
Let $Q = M \triangle M'$
Q has more edges from M' than M.
Each vertex in V is incident to at most one edge in $M \cap Q$ and at most one edge in $M' \cap Q$, by the fact that M and M' are matchings, thus, the subgraph Q has degree 2.
Q is the union of paths and cycles that alternate between M and M'.

All Cycles in Q have even length

$\exists$ a path in Q with more edges in M' than in M

This path is augmenting w.r.t. M. $\square$



3 Algorithm

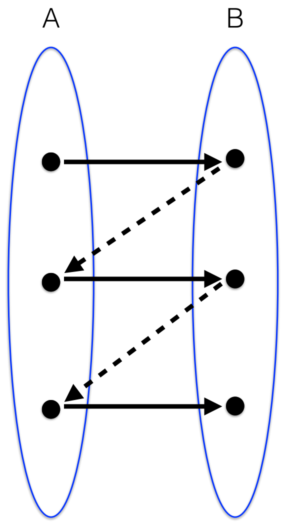
1. $M = \Phi$
2. While $\exists$ augmenting path P w.r.t. M
   - $M = M \triangle P$
3. end

Number of Iterations of this algorithm is at most $\frac{n}{2}$

4 Finding an Augmenting Path in a bipartite Graph

Construct new directed graph D by orienting edges G as follows:

- if $e \notin M$, orient from A to B.
- if $e \in M$ orient from B to A.



Lemma 4.1. *$\exists$ augmenting path w.r.t. M in G if and only if $\exists$ a directed path in D from an exposed vertex in A to an exposed vertex in B.*