Maximum Matching (Non-bipartite)

Definition: Flower (w.r.t. matching $M$)

\[
\text{Algorithm:} \\
M = \emptyset \\
\text{Repeat until there are no augmenting paths and no flowers:} \\
\text{if } \exists \text{ an augmenting path } P, \text{ let } M = M \cup P \\
\text{if } \exists \text{ flower } F \text{ then:} \\
\quad \text{let } Q \text{ be the stem of } F, \text{ and } B \text{ be the blossom of } F. \\
\quad \text{Let } M = M \cup Q \\
\quad \text{let } G = G/B \text{ (contract } B \text{ into single vertex)} \\
\text{end} \\
\text{end}
\]

Lemma: $M$ is a maximum matching in $G$ iff $M/B$ is a maximum matching in $G/B$.

Proof: ($\Rightarrow$) Let $N$ be a matching in $G/B$ bigger than $M/B$. $N$ is incident to at most one vertex of $B$ in $G$ (after pulling back). Extend $N$ to a matching $N'$ in $G$ by adding $\frac{1}{2}(|B|-1)$ edges in $B$. Thus $|N'| - |M| = |N| - |M/B|$.

Thus $|N'| > |M|.$

($\Leftarrow$) If $M$ is not maximum there $\exists$ augmenting path $P$ (w.r.t. $M$) in $G$ with endpoints $u$ and $v$. $B$ has only one exposed vertex, thus we may assume wlog that $u \notin B$. If $PB = \emptyset$ then $P$ is also augmenting in $G/B$. Otherwise let $w$ be the first vertex of $P$ in $B$. 
Let $Q$ be the subpath of $P$ from $u$ to $w$.

Note that $b$ (the vertex into which we contract $B$) is exposed in $G/B$.

Thus $Q$ is augmenting in $G/B$.

**Corollary:** The algorithm is correct.

How to find an augmenting path or a flower:

**Alternating tree:**

```
Alternating tree:   x  exposed  even
                   \           odd
                    even
                   \           odd
                    even
```

Start "growing" an alternating tree rooted at each exposed node.

All exposed nodes are marked "even".

**Processing some even vertex $u$:**

- **Case 1:** $E(u,v) \in E$ with $v$ unlabelled.
  - Label $v$ "odd". $v$ is not exposed (or it would have been labeled even).
  - Label its "match" $w$ "even".
Case 2: \( \exists (u,v) \in E \) with \( v \) labelled "even".
Then \( v \) belongs to another alternating tree. We have found an augmenting path:

```
exposed
aug. path
```

Case 3: \( \exists (u,v) \) with \( v \) labelled "even" and \( v \) belongs to the same alternating tree as \( u \).
Then we have found a flower:

```
\text{not (exposed)}
```