1 Introduction

These notes continue to describe how to apply Baker's technique in order to obtain a polynomial time approximation algorithm for the Independent Set problem. As a reminder, we proved the following lemma in a previous lecture, where each G_i^j is a component of a graph obtained by Baker's technique.

Lemma 1. *For all i, j, G_i^j is $O(k)$-outerplanar.*

Additionally, we previously proved the following theorem, which bounds the treewidth of a graph by the previously described "vertex remember number" and "edge remember number".

Theorem 1. *Let $G = (V, E)$ and T be a maximal spanning forest (MSF) of G. Then the treewidth $tw(G) = \max\{vr(G, T), er(G, T) + 1\}$, where $vr(G, T)$ denotes the "vertex remember number" of G with respect to T and $er(G, T)$ denotes the "edge remember number" of G with respect to T.*

To finish the application of Baker's technique to the Independent Set problem, we will combine Lemma 1 and Theorem 1 in order to show how we can find a polynomial-time approximation algorithm for the Independent Set problem.

As an overview for the rest of these notes, we will focus on showing bounds on the edge remember number and vertex remember numbers of a k-outerplanar graph. We will then use these bounds to show that the treewidth of a k-outerplanar graph is in $O(k)$. We can then use dynamic programming on the tree decompositions of the components G_i^j to approximate a solution in polynomial time once such a bound is put on the treewidth.

2 Bounding Edge and Vertex Remember Numbers

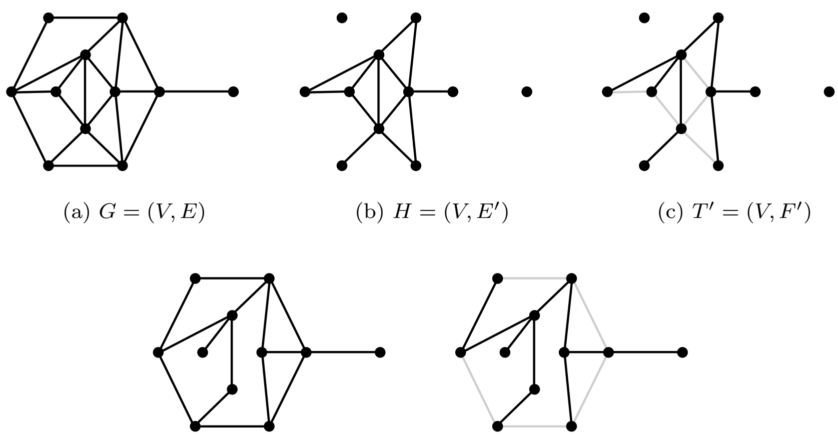
First, we describe bounds on the edge and vertex remember numbers of planar graphs.

Lemma 2. *Let $G = (V, E)$ be planar, with a fixed planar embedding. Let $H = (V, E')$ be the graph from G obtained by deleting all edges in the outer face. Let $T' = (V, E')$ be a MSF of H. Then there exists a MSF $T = (V, F)$ of G with:*

$$er(G, T) \leq er(H, T') + 2 \text{ and } vr(G, T) \leq vr(H, T') + degree(G).$$

Proof of Lemma 2. We first describe how to construct the MSF T mentioned in Lemma 2. Let the graphs $G = (V, E)$, $H = (V, E')$, and $T' = (V, F')$ be constructed as described in Lemma 2. Now let $K = (V, (E \setminus E') \cup F')$ and $T = (V, F)$ be a MSF of K with the constraint that $T' \subseteq T$.

It follows from this construction that T is also a MSF of G. It can be seen that T is also a MSF of G because two vertices are in the same connected component of K if and only if they are in the same connected component of G. So when a MSF T of K is constructed in the decsribed manner, T is also a MSF of G. Sample illustrations of the previously described graphs are shown below in Figure 1 for clarity.



(a) $G = (V, E)$ (b) $H = (V, E')$ (c) $T' = (V, F')$

(d) $K = (V, (E \setminus E') \cup F')$ (e) $T = (V, F)$

Figure 1: Example constructions of the graphs described in Lemma 2. (a) The original graph G. (b) The graph H, which is constructed by removing the edges in the outer face of G. (c) A MSF T' of H. (d) The graph K used to constructed a MSF of G. (e) A MSF T of both K and G. Note that the gray edges in (c) and (e) are the edges in the graph that the MSF is constructed from, but not in the MSF itself.

Now, we must show that T obeys the inequalities decribed in Lemma 2. Observe the faces of K. Each new fundamental cycle introduced by the edges of the outer face of G and T is precisely a face of K. No edge of K can be included in more than two faces of K, so each edge can at most be included in at most two additional fundamental cycles after the addition of the edges of the outer face of G. It follows that $er(G, T)$ is at most two greater than $er(H, T')$. Therefore, we obtain the inequality:

$$er(G, T) \leq er(H, T') + 2.$$

Similarly, a vertex of G cannot be included in more than more than $degree(G)$ faces, so each vertex can be included in at most $degree(G)$ additional fundamental cycles after the addition of the edges of the outer face of G, and $vr(G, T)$

is at most $degree(G)$ greater than $er(H, T')$. We now obtain the inequality:

$$vr(G, T) \leq vr(H, T') + degree(G).$$

$\square$

We cannot yet use Lemma 2 to provide a bound on the edge and vertex remember numbers that only depends on k for a k-outerplanar graph because our current bounds currently use the degree of the graph. We begin simplifying this bound by first considering outerplanar graphs with bounded degree and then k-outerplanar graphs with bounded degree.

Lemma 3. *Let $G = (V, E)$ be an outerplanar graph with $degree(G) \leq 3$. Then there exists a MSF T of G with $er(G, T) \leq 2$ and $vr(G, T) \leq 3$.*

Proof of Lemma 3. Given such an outerplanar graph $G = (V, E)$ with $degree(G) \leq 3$, remove all edges in the outer face of G to obtain a forest $T' = (V, F')$. Trivially, we have $er(T', T') = vr(T', T') = 0$. Now using Lemma 2, we easily obtain the inequalities:

$$er(G, T) \leq 2 \text{ and } vr(G, T) \leq degree(G) = 3.$$

$\square$

More generally, we can show the following:

Lemma 4. *Let $G = (V, E)$ be a k-outerplanar graph with $degree(G) \leq 3$. Then there exists a MSF $T = (V, F)$ of G with $er(G, T) \leq 2k$ and $vr(G, T) \leq 3k$.*

Proof of Lemma 4. Consider a k-outerplanar graph $G = (V, E)$ with $degree(G)$ at most three. From Lemma 3 we know that if $k = 1$, then there exists a MSF T such that $er(G, T) \leq 2$ and $vr(G, T) \leq 3$. From Lemma 2 we can see that each layer introduced in a planar graph will at most increase the edge remember number by at most two and will increase the vertex remember number by at most the degree of the graph. But since the $degree(G) \leq 3$, the vertex remember number can only increase by at most three at each layer for such a G. It now easily follows from induction that for such a graph G, there exists a MSF T such that:

$$er(G, T) \leq 2k \text{ and } vr(G, T) \leq 3 \cdot degree(G) = 3k.$$

$\square$

3 Bounding Treewidth

Although we cannot always assume that our k-outerplanar graph will have a degree of at most three, we can introduce the following lemma to supply a method of combining our previous theorems and lemmas together to bound the treewidth of a k-outerplanar graph.

Lemma 5. *For all k-outerplanar $G = (V, E)$, there exists a k-outerplanar $H = (V', E')$ such that $G \preceq H$ and $degree(H) \leq 3$.*

Proof of Lemma 5. Given a k-outerplanar $G = (V, E)$, a graph with the previously described properties H can always be constructed by replacing high degree vertices with paths of vertices. More specifically, for any vertex in $G(V)$ with a degree n greater than three, it can be replaced by a path of $n - 2$ vertices. The original edges in $G(E)$ can then be added in such a way to ensure that the degree of a vertex never exceeds three and that G is a minor of H. This construction process also maintains the number of layers in the planar graph. An example of such a construction is shown in Figure 2.

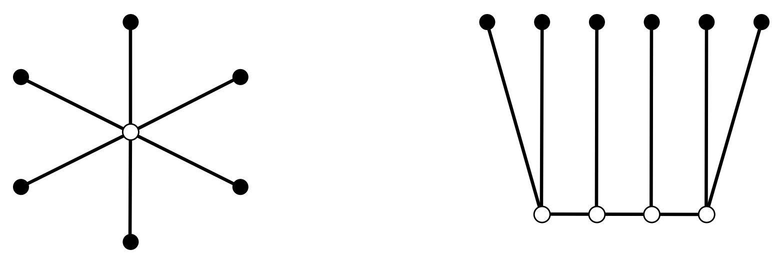


Figure 2: The center white vertex in the graph on the left has degree six. It is replaced by the path of the four white vertices in the graph on the right. See that the graph on the right has a degree of three.

$\square$

We are now able to combine our previous results to provide a bound in $O(k)$ on the treewidth of a k-outerplanar graph.

Lemma 6. *For all k-outerplanar graphs G, we have $tw(G) \leq 3k$.*

Proof of Lemma 6. Let $G = (V, E)$ be k-outerplanar. As shown in Lemma 5, we can construct a k-outerplanar H, with $degree(H) \leq 3$ and $G \preceq H$. From Lemma 4, there exists a MSF T of H such that $vr(H, T) \leq 3k$ and $er(H, T) \leq 2k$. The treewidth of H can be bounded by Theorem 1 and the inequality $tw(H) \leq \max\{2k + 1, 3k\} = 3k$ is obtained. Lastly, since $G \preceq H$, it follows that $tw(G) \leq tw(H) \leq 3k$ and the treewidth of G is in $O(k)$. $\square$

For any k-outerplanar graph G, we have now shown that its treewidth is in $O(k)$. Since the components G_i^j obtained through Baker's technique are k-outerplanar, we know that their treewidths are also in $O(k)$. Therefore, dynamic programming can be used on the tree decompositions of each G_i^j to solve the Independent Set problem for each G_i^j in polynomial time. Using the algorithm described in the first lecture on Baker's theorem, we are then able to use the Independent Set solutions for each G_i^j in order to obtain a polynomial time approximation algorithm for the Independent Set problem for fixed k.

4 An Additional Application of Baker's Technique

Baker's technique can be applied to a number of problems to obtain polynomial time approximation algorithms. One such problem is Vertex Cover.

Vertex Cover

Input: $G = (V, E)$

Goal: Find $X \subseteq V$, such that for all $\{u, v\} \in E$, $\{u, v\} \cap X \neq \emptyset$, minimizing $|X|$.

Recall that Baker's technique for Independent Set solves the Independent Set problem on small, non-overlapping slices of the input graph. We cannot use such slices for Vertex Cover as it is possible an edge will be missed in a region that the slices do not cover. Informally, Baker's technique is applied to Vertex Cover by computing overlapping slices of the graph and then solving Vertex Cover for those slices. Although the solution will not be minimal, the overlapping slices ensure that the all edges are properly covered. This technique also approximates the solution of Vertex Cover up to a $(1 + 1/k)$ factor.