Problem Set 4
TTIC 31100 / CMSC 39000 Computational geometry

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Recall that the *edge expansion* $\alpha(S)$ of a set of vertices $S \subset V$ equals $|\delta(S, V \setminus S)|/|S|$, and the expansion of a graph G equals

$$h_G = \min_{\substack{S \subset V \\ 0<|S|\leq|V|/2}} \alpha(S).$$

A graph G is r-regular if every vertex of G has degree r. We denote the shortest path distance in G by d_G.

Problem 1. The goal of this exercise is to prove that that there exists a family of metric spaces X_n on n points that require distortion $\Omega(\log n)$ for embedding into ℓ_1.

1. Suppose that $G = (V, E)$ is a r-regular graph. Prove that for every vertex u and a random vertex v,

$$\Pr_{v \in V}(d_G(u, v) \leq t) \leq \frac{r^{t+1} - 1}{(r-1)n}.$$

   Conclude that if G is a 3-regular graph, $\Pr_{v \in V}(d_G(u, v) \geq \lfloor \log_3 n \rfloor) \geq 1/2$.

2. Suppose that $G = (V, E)$ is a 3-regular graph. Prove that

$$\frac{1}{|V \times V|} \sum_{u,v \in V \times V} d_G(u, v) \geq \frac{\lfloor \log_3 n \rfloor}{2} = \frac{\lfloor \log_3 n \rfloor}{2} \cdot \frac{1}{|E|} \sum_{\{u,v\} \in E} d_G(u, v).$$

3. Suppose that $G = (V, E)$ is a 3-regular graph and S is a subset of V, $0 < |S| \leq |V|/2$. Denote the cut metric corresponding to the cut $(S, V \setminus S)$ by $\delta_S(\cdot, \cdot)$. Prove that

$$\frac{h_G}{3} \cdot \frac{1}{|V| \times |V|} \sum_{u,v \in V \times V} \delta_S(u, v) \leq \frac{2h_G|S|}{3n} \leq \frac{1}{|E|} \sum_{\{u,v\} \in E} \delta_S(u, v).$$

4. Conclude that every embedding of the metric space (V, d_G) into ℓ_1 requires distortion at least $\frac{h_G \lfloor \log_3 n \rfloor}{6}$.

It is known that there exists a 3-regular graph $G_n = (V_n, E_n)$ on n vertices with $h_{G_n} \geq c$ for every even n, where $c > 0$ is some absolute constant[1]. It follows from item 4, that metric spaces (V_n, d_{G_n}) asymptotically require distortion $\Omega(\log n)$ for embedding into ℓ_1.

[1] Such graphs with constant expansion h_G are called expanders. We require that n is even since there are no 3-regular graphs with odd number of vertices.

Problem 2. In the Minimum Linear Arrangement problem, given a graph $G = (V, E)$, we need to find a one-to-one mapping φ from V to $\{1, 2, \ldots, n\}$ that minimizes the following objective function

$$\text{cost}(\varphi) = \sum_{(u,v)\in E} |\varphi(u) - \varphi(v)|.$$

1. Consider a feasible solution φ. Let $S_k = \{u : \varphi(u) \leq k\}$. Prove that

$$\text{cost}(\varphi) \geq \sum_{i=1}^{n-1} |\delta(S_i, V \setminus S_i)|.$$

2. Let $\text{OPT-BalCut}_{1/3}(G)$ be the size of the smallest 1/3–balanced cut in G. Prove that $\text{cost}(\varphi) \geq (n/3 - 2)\text{OPT-BalCut}_{1/3}(G)$.

3. Consider the following recursive algorithms.

function *LinearArrangement*
    Input: a graph $H = (U, E_H)$;
    Output: a one-to-one map $\varphi : U \to \{1, \ldots, |S|\}$.
begin
    if U contains two or less vertices **return** any feasible mapping.
    find a 1/6–balanced cut S of value $O(\log |H|) \cdot \text{OPT-BalCut}_{1/3}(H)$
    $\varphi_1 = \mathit{LinearArrangement}(H[S])$;
    $\varphi_2 = \mathit{LinearArrangement}(H[V \setminus S])$;
    return the concatenation φ of arrangements φ_1 and φ_2
    (that is, $\varphi(u) = \varphi_1(u)$ if $u \in S$; $\varphi(u) = \varphi_2(u) + |S|$ if $u \notin S$)
end

Prove that it finds a linear arrangement of cost $O(\log^2 n)\text{OPT}$, where OPT is the cost of the optimal linear arrangement.

Problem 3. Using the fact that every n-point metric stochastically embeds into a distribution over trees with distortion $O(\log n)$, give an alternative proof for the fact that every n-point metric embeds into ℓ_1, with distortion $O(\log n)$. Hint: Use the fact that every tree embeds into ℓ_1 isometrically, i.e. with distortion 1.

Problem 4. Let S^2 denote the 2-dimensional unit sphere in $\mathbb{R}^3$, i.e.

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Show that the metric space $(S^2, \|\cdot\|_2)$ stochastically embeds into the Euclidean plane, with distortion $O(1)$.