On graph crossing number and edge planarization

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Computational topology

- Computation on topological objects
  - Algorithms on topologically simple input
  - Recognition of topological invariants
- Dimensionality
  - dim = 1: Trivial
  - dim = 2: Most of known computational topology. (planar graphs, graphs on surfaces, 2-manifolds, etc)
  - dim = 3: Most problems open. (knots, recognition of 3-manifolds, etc)
  - Higher dimensions: most problems intractable / undecidable

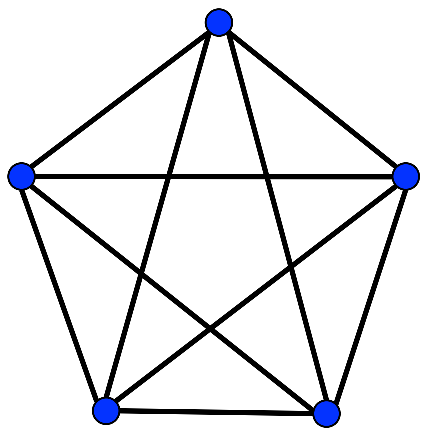
2-dimensional topological invariants

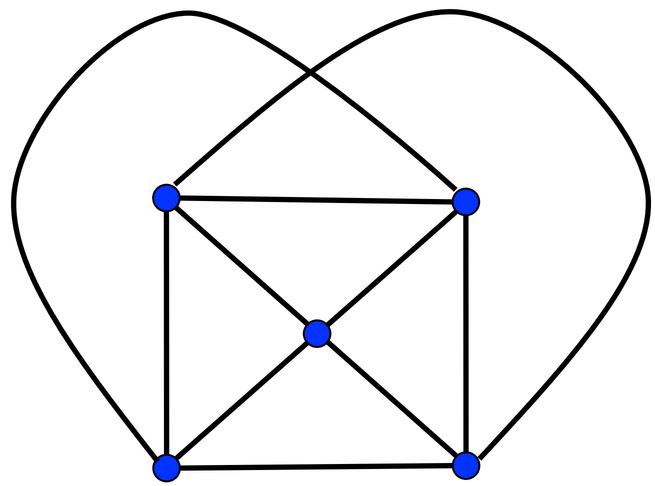
- Planarity
- Genus
- Crossing number
- Min edge/vertex planarization

Recognition of topological invariants

- Exact algorithms
  - Planarity testing [Hopcroft,Tarjan’74]
  - Genus [Mohar’99], [Kawarabayashi,Mohar,Reed’08]
  - Graph minor theorem [Robertson,Seymour’99]
  - Crossing Number [Kawarabayashi,Reed’07], [Grohe’04]
- Approximation algorithms
  - Partial results on Crossing Number

Crossing number





$cr(K_5) = 1$

$cr(K_{13}) = ?$

Computing the crossing number

- **> 600** papers on Crossing Number [Vrt'o]
- No PTAS for *cr(G)*. [Garey,Johnson'83], [Ambuhl,Mastrolilli,Svensson'07]
- Linear-time algorithm for fixed *cr(G)*. [Kawarabayashi,Reed'07], [Grohe'04]
- $O(n \cdot \log^2 n)$-approximation [Even,Guha,Schieber'02], [Leighton,Rao'92], [Bhatt,Leighton'84], [Arora,Rao,Vazirani'09]
- Better approximations for special classes
  - Planar + one edge: NP-hard [Cabello,Mohar'10], $O(1)$-approx. [Hlineny,Salazar'07]
  - 1-apex graphs: $O(1)$-approx. [Chimani,Hlineny,Mutzel'09]
  - Toroidal graphs: $O(1)$-approx. [Hlineny,Salazar'07]
  - Projective graphs [Gitler,Hlineny,Leanos,Salazar'07]
  - Some small genus graphs [Hlineny,Chimani'10]

Our results

- **Theorem** [Chuzhoy, Makarychev, S ’11]
  Given a graph G = planar + k edges,
  we can find a drawing with $O(k \cdot (k+cr(G)))$ crossings.

- **Corollary** [Chuzhoy, Makarychev, S ’11]
  $O(n \cdot \log^{3/2} n)$-approximation for $cr(G)$.

- **Corollary** [Chuzhoy, Makarychev, S ’11]
  $2^{O(g)} \cdot n^{1/2}$-approximation for $cr(G)$ on genus-g graphs.

- **Corollary** [Chuzhoy, Makarychev, S ‘11]
  $O(1)$-approximation for $O(1)$-apex graphs.

Further implications

- Every graph can be made planar by removing *cr(G)* edges.

- **Corollary** [Chuzhoy, Makarychev, S '11]
  Any approximation for Min-Edge-Planarization implies an approximation for *cr(G)*.

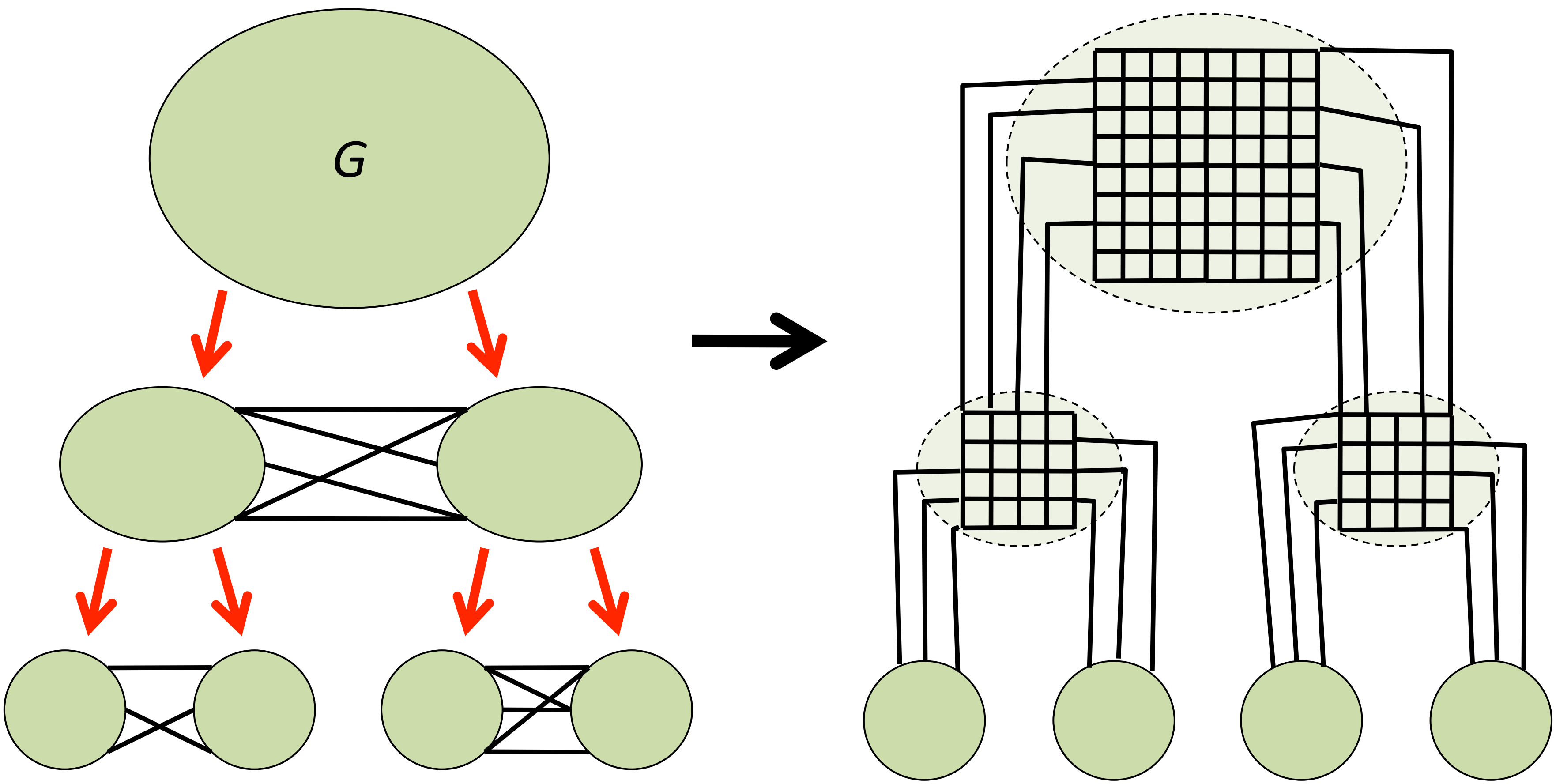
- **Theorem** [Chuzhoy]
  $OPT^{O(1)}$-approximation Min-Edge-Planarization.

- **Corollary**
  $O(n^{1-\alpha})$-approximation for *cr(G)*.

$O(n \cdot \log^2 n)$-approximation for $cr(G)$

- **Main idea:** [Bhatt,Leighton'84], [Leighton,Rao'92], [Even,Guha,Schieber'02]
  - Planar graphs have small separators. [Lipton,Tarjan'79]
  - Small crossing number implies small separators.
  - Divide & conquer.

$O(n \cdot \log^2 n)$-approximation for $cr(G)$



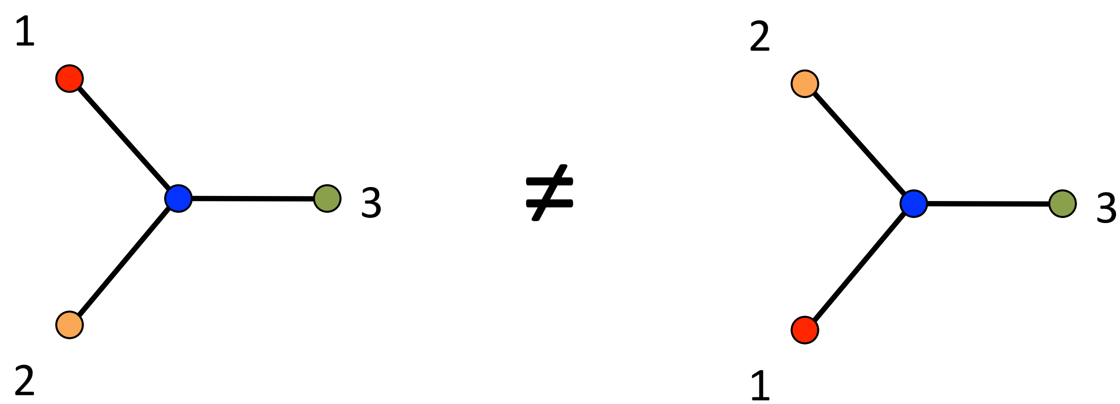
Our approach

Key idea*:*

Any drawing of a 3-connected graph with a few crossings, is ``close'' to a planar drawing.

Rotation systems

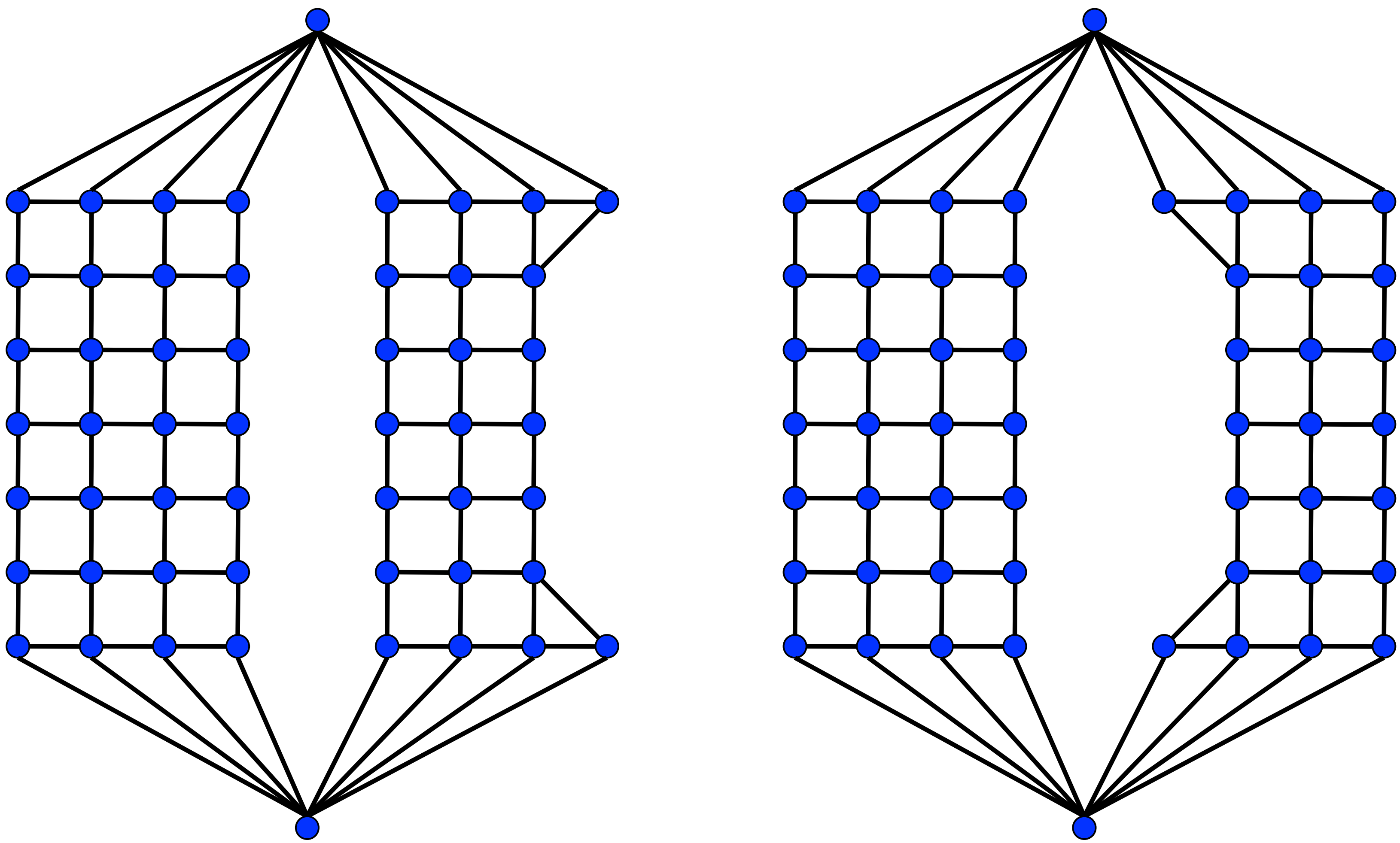
- For every vertex, specify the cyclic ordering of its adjacent edges



Local vs global

- 3-connected planar graphs, have a unique drawing. [Whitney ‘32]
- All planar drawings of a 3-connected planar graph have the same rotation system, up to orientation.

Example



Approximate rotation systems

- What about non-planar drawings?
  - Do all drawings with a few crossings have “similar” rotation systems?
  - For drawings φ, ψ let $IRG(\varphi,\psi)$ be the set of vertices with different orderings in the corresponding rotation systems.

Lemma [Chuzhoy, Makarychev, S ’11]

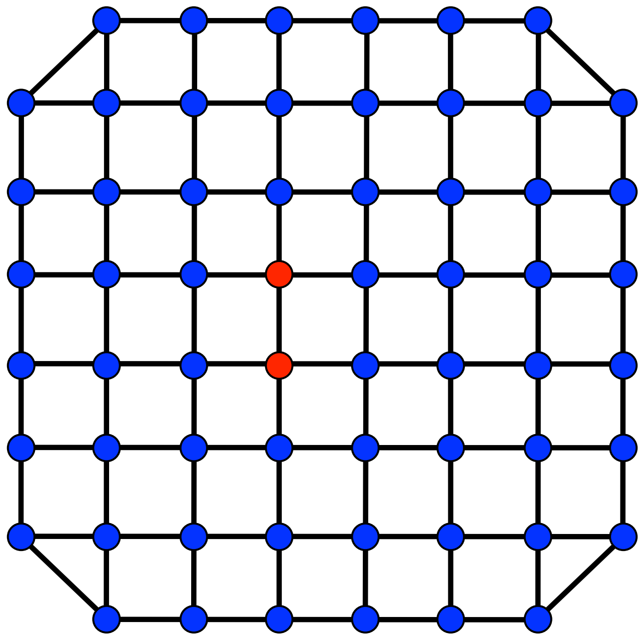
Let G be a 3-connected planar graph.

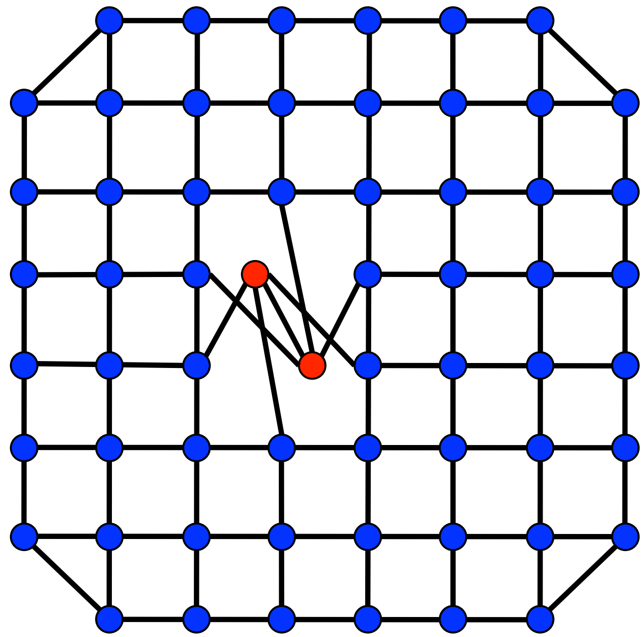
Let φ be the unique planar drawing of G.

Let ψ be a drawing of G with s crossings.

Then, $IRG(\varphi,\psi) = O(s)$.

Approximate rotation systems (example)

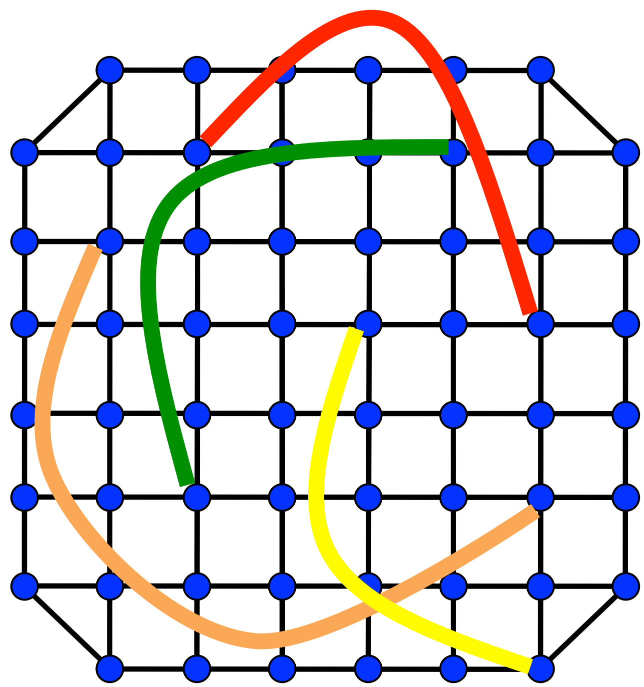




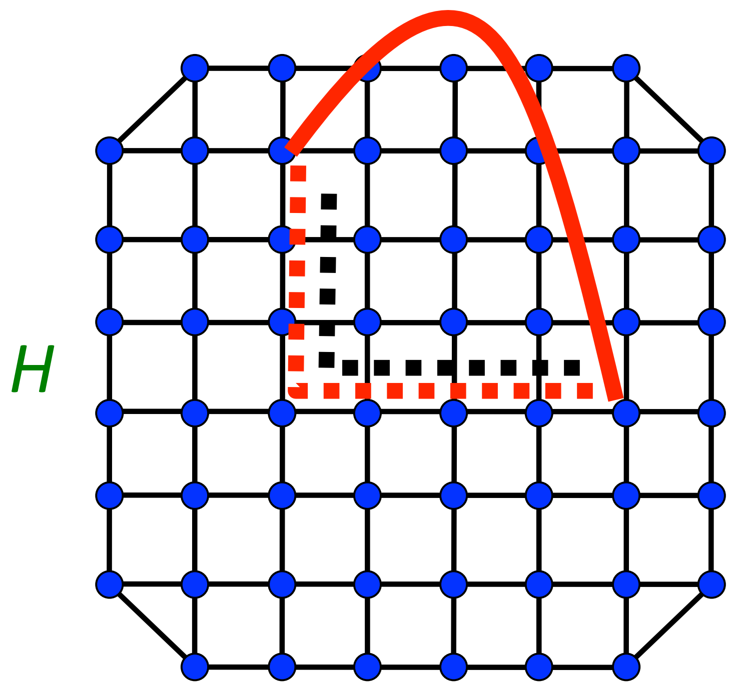
The main argument

G = planar *H* + *k* edges.

Assume *H* is **3-connected**.



Routing along paths



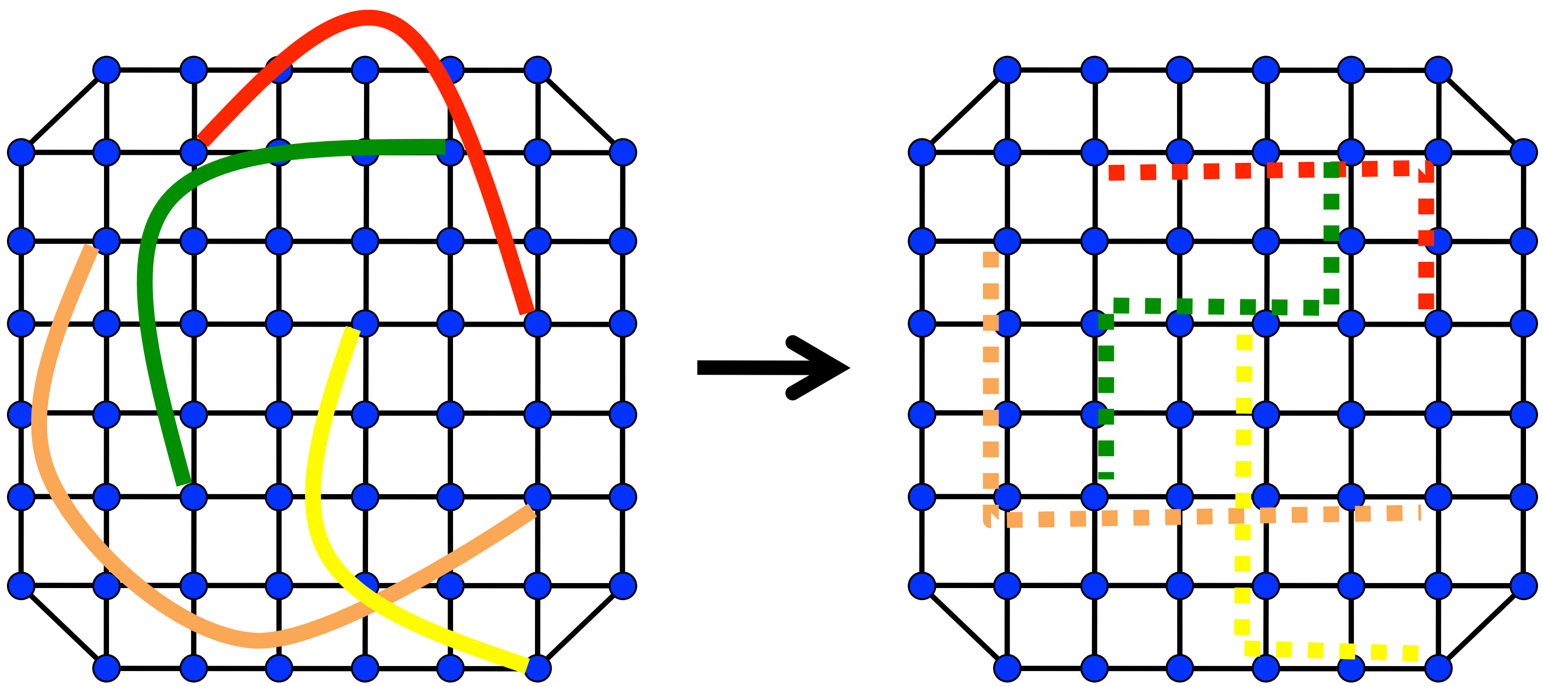
Draw every extra edge "close" to a shortest path in the dual graph.

The main argument (cont.)

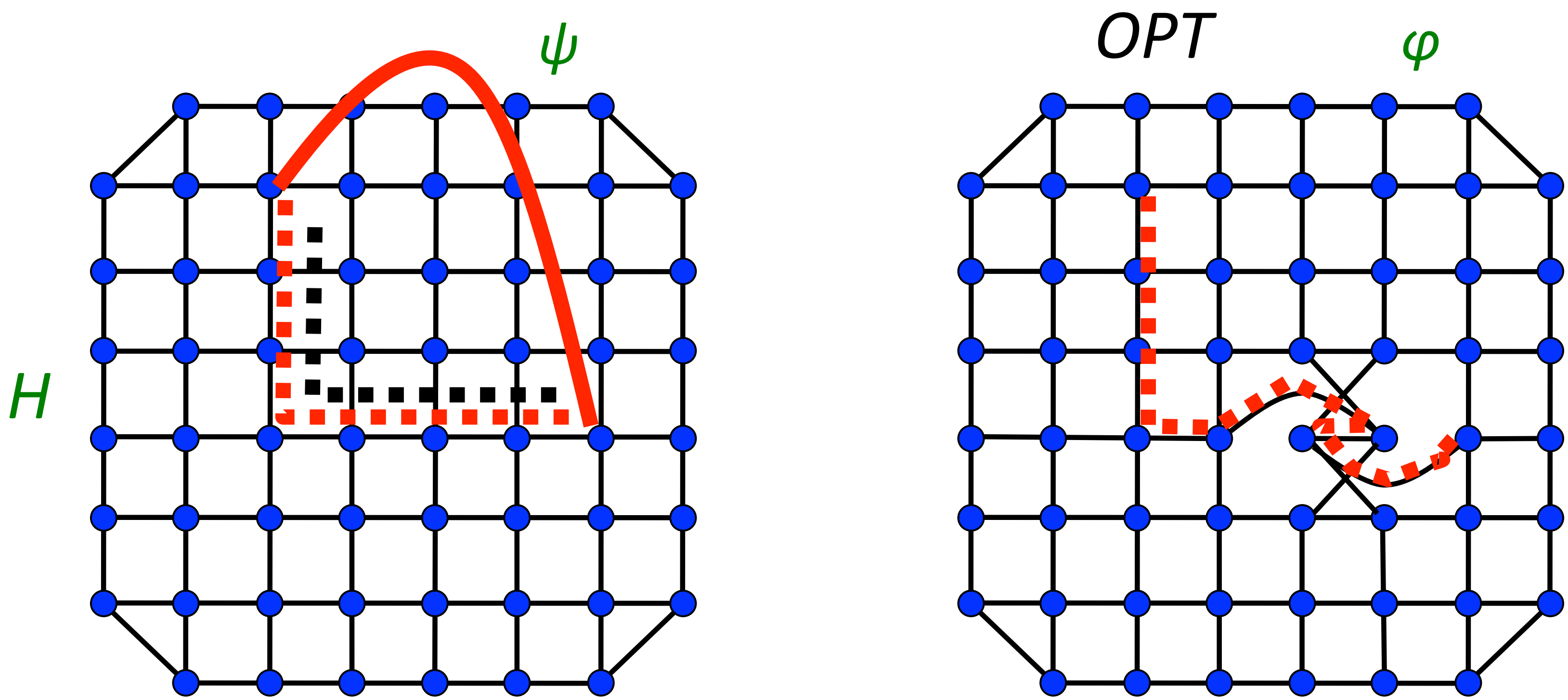
Route extra edges along paths of *H*.

$O(k^2)$ crossings between the routing paths.

Suffices to bound crossings between routing paths and *H*.



The main argument (cont.)



Lemma:

Every routing path has at most *O(cr(G))* more crossings in *ψ* compared to *OPT*.

The general case

- *H* might not be 3-connected
- We need to find a planar drawing of *H*, with a rotation system similar to the optimal.
- Block decomposition, similar to **SPQR trees**.
- Many technical details…

Further directions

- *G = planar + k edges*, is there an *O(k)*-approx. for *cr(G)*?
- *O(1)*-approx. for *cr(G)*?
- Other topological parameters?
- Approximate versions of the Graph Minor Theorem?
- We know a few separations.
  - Clique minors: $\Omega(n^{1/2})$-hard [Alon,Lingas,Wahlen'07], fixed-parameter tracktable [Robertson,Seymour'86]