Probabilistic Embeddings of Bounded Genus Graphs Into Planar Graphs

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Probabilistic Embeddings

- Given finite metric space $M=(X,D)$
- Obtain distribution $F=\{M_1,M_2,\ldots,M_k\}$, $M_i=(X,D_i)$, such that $\forall$ u,v $\in X$,
  - $\forall$ $M_i \in F$, $D_i(u,v) \geq D(u,v)$
  - $\mathbf{E}_{N \in F}[D_N(u,v)] \leq \alpha \cdot D(u,v)$

α : distortion
GOAL : small α

Probabilistic Embeddings - Known Results

From	Into	Upper	Lower	Citation
Cycle	Line	$O(1)$	$\Omega(1)$	[Karp89]
General	Trees	$O(\log n)$	$\Omega(\log n)$	[Alon,Karp,Peleg,West'91], [Bartal'96], [Bartal'98], [Fakcharoenphol,Rao,Talwar'03]
General Graphs	Subtrees	$O(\log^2 n \log\log n)$	$\Omega(\log n)$	[Elkin,Emek,Spielman,Teng'05]
Series-Parallel	Subtrees	$O(\log n)$	$\Omega(\log n)$	[Emek,Peleg'06]
Doubling	Small Treewidth	$1+\varepsilon$		[Talwar'04]
Treewidth-k	Treewidth-(k-3)	$O(\log n)$	$\Omega(\log n)$	[Carroll,Goel'04]
O(1)-Genus	Planar	$O(1)$	$\Omega(1)$	[Indyk,S'06]

Implications

Approximation algorithms:
Let *A* be an optimization problem, s.t. the objective depends linearly on the distances of the input metric.
(e.g. Shortest-Paths, MST, k-Median, Clustering, etc.)

If there exists an *a*-approximation for *A* on planar graphs, then there exists an *O(a)*-approximation for *A* on bounded-genus graphs.

Embedding into L_1:
If all planar graphs embed into L_1 with distortion γ, then all bounded-genus graphs embed into L_1 with distortion $O(\gamma)$.

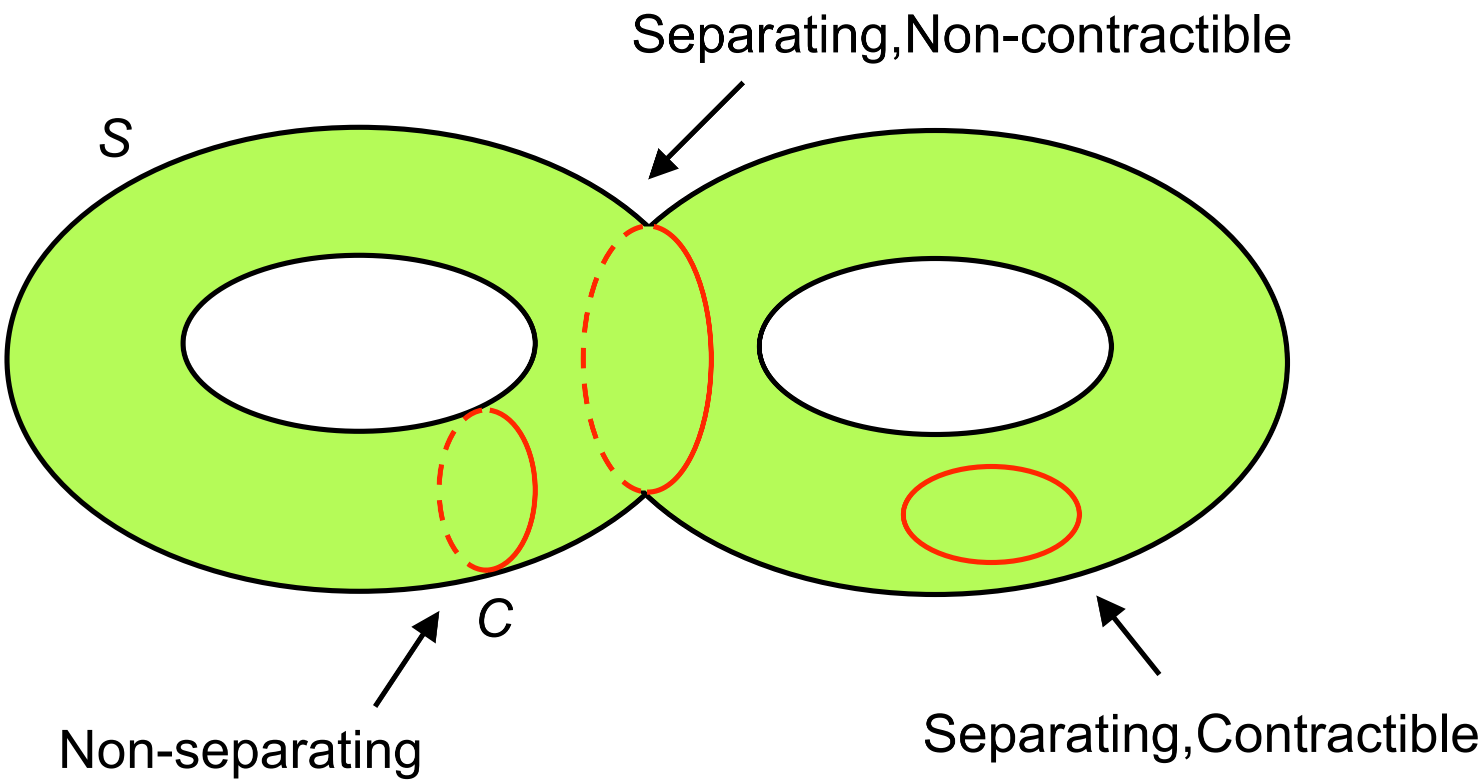
Deterministic Embeddings?

There exists a graph of genus *1*, s.t. any deterministic embedding into a planar graph has distortion $\Omega(n)$.

Using arguments similar to [Rabinovich,Raz'98], [Gupta'01], [Matousek], [Carroll,Goel'04]

Thus, **randomization is necessary**.

Curves on Orientable Surfaces - Crash Course



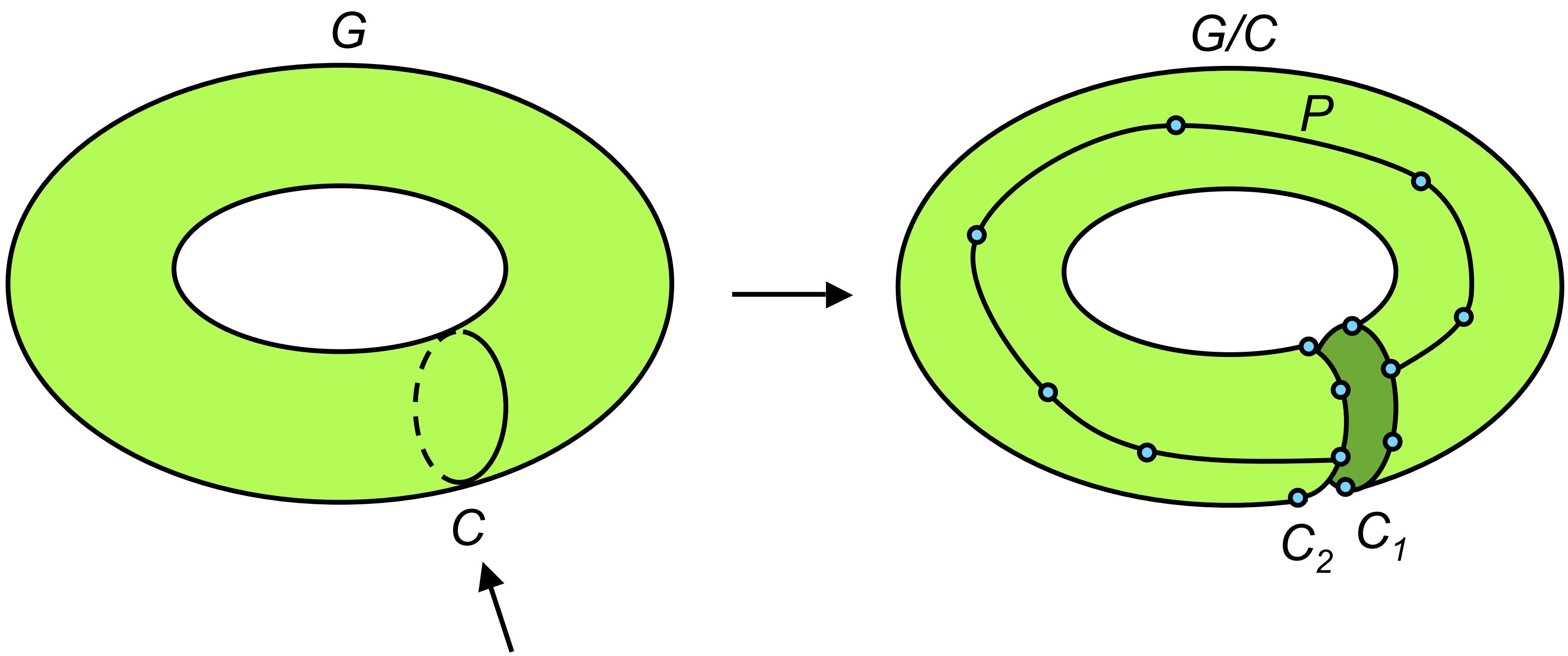
Fact: *genus(S\C) = genus(S) - 1*

Planarization

Planarization Algorithm:

- Find non-separating cycle *C*
- Remove *C*
- Repeat until planar

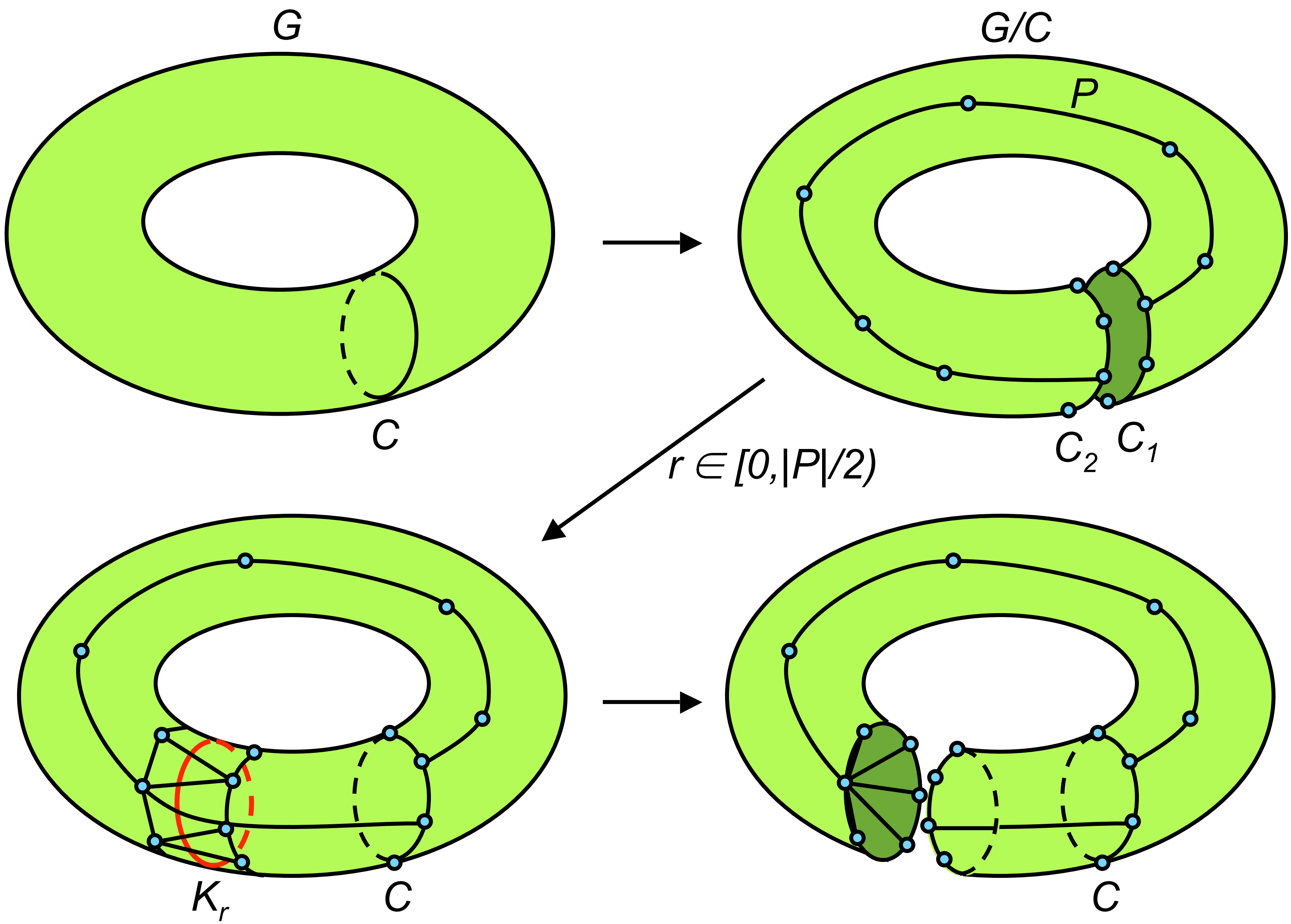
Reducing the genus by 1



Shortest non-separating cycle
Can be computed e.g. by [Cabello,Chambers'07]

Claim: $|P| \geq |C|/2$

Reducing the genus by 1



Analysis

Consider edge $e=\{u,v\}$

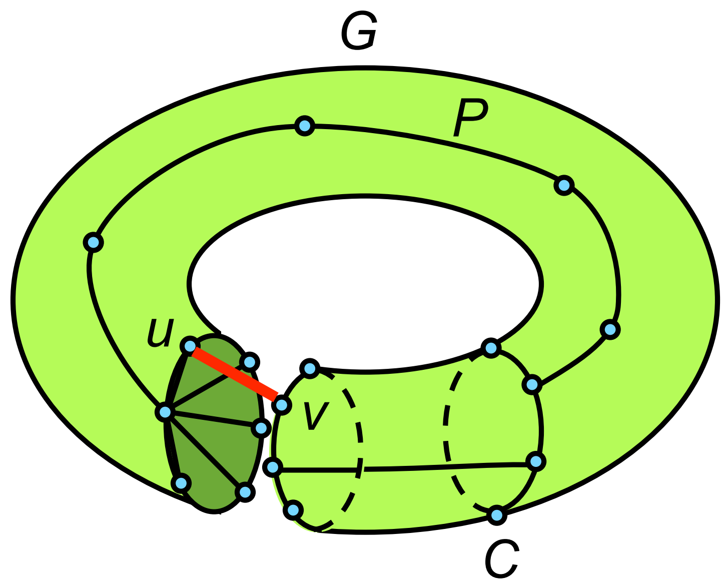
- $Pr[e$ is cut$] \leq 2D(u,v) / |P|$
- If e is cut, then

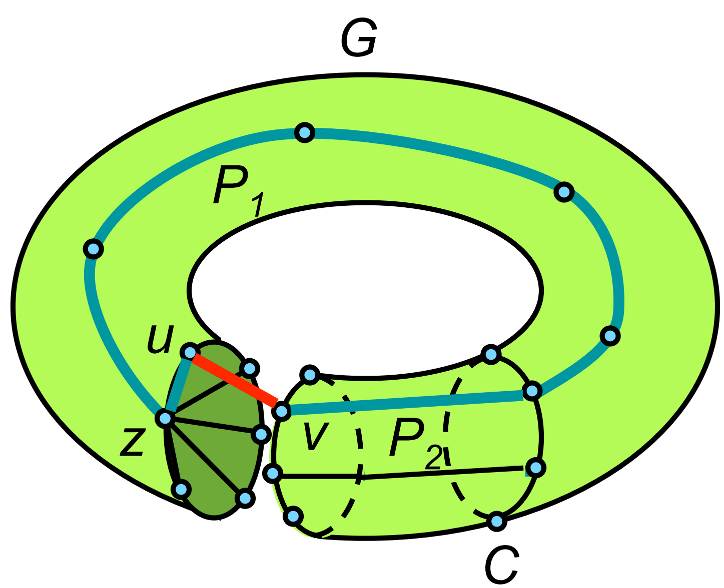
$$D'(u,v) \leq D'(u,z) + |P_1| + |P_2|$$
$$\leq D(u,z) + |P| + |P|$$
$$\leq 2|P| + |P|/2 + |P|/2$$
$$= O(|P|)$$

- Thus,

$$E[D'(u,v)] = D(u,v)\ Pr[e \text{ not cut}] + O(|P|)\ Pr[e \text{ is cut}] = O(D(u,v))$$

For arbitrary paths, apply linearity of expectation.





Conclusions

- After repeating g times, distortion = $2^{O(g)}$

- Lower bound $\Omega(\log(g)/\log\log(g))$

  Using standard counting argument

Question: Can we do better?

- Treewidth-6 graphs into planar graphs, $\Omega(\log(n))$ [Carroll,Goel'04].

  Thus, there is no generalization to arbitrary minor-closed families!