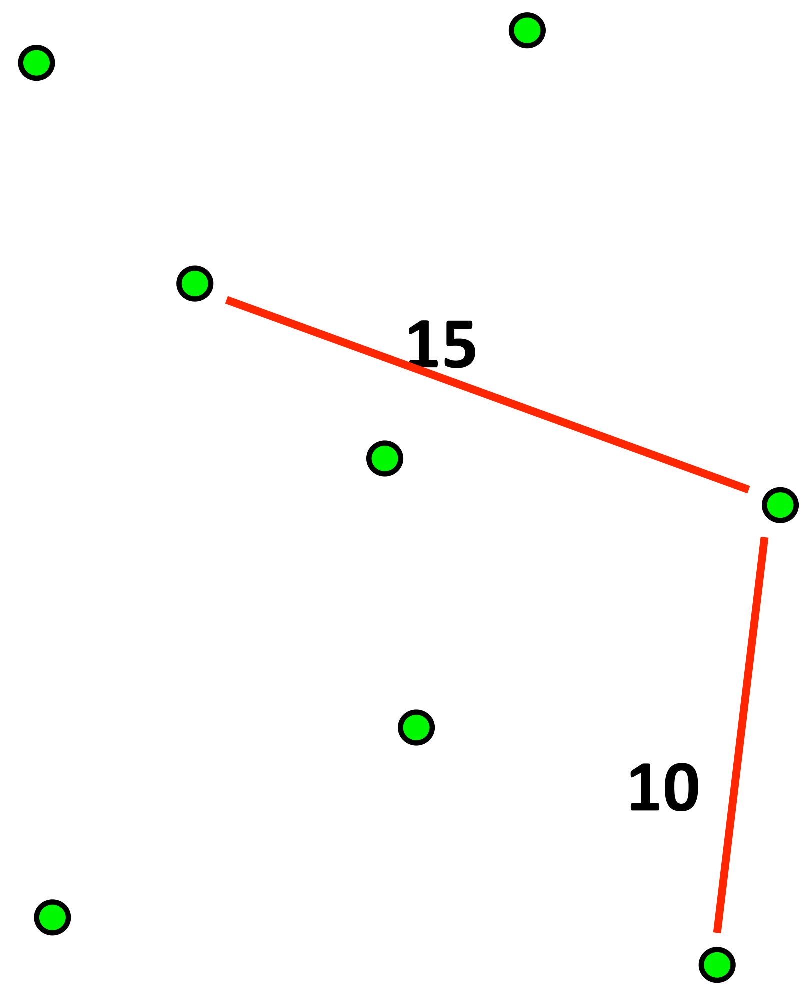
Genus and the Geometry of the Cut Graph

James R. Lee (University of Washington)
Anastasios Sidiropoulos (Toyota Technological Institute)

Metric spaces

Metric space M=(X,d)

- Positive definiteness
  $d(p,q) = 0$ iff $p = q$
- Symmetry
  $d(p,q) = d(q,p)$
- Triangle inequality
  $d(p,q) \leq d(p,r) + d(r,q)$

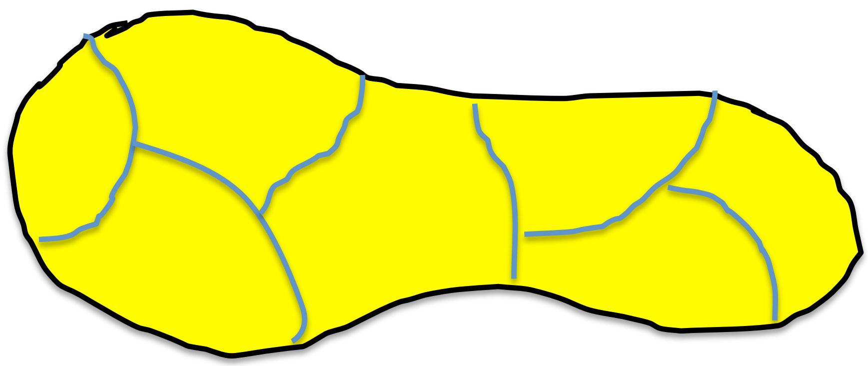


Equivalently:

Shortest-path metrics on graphs

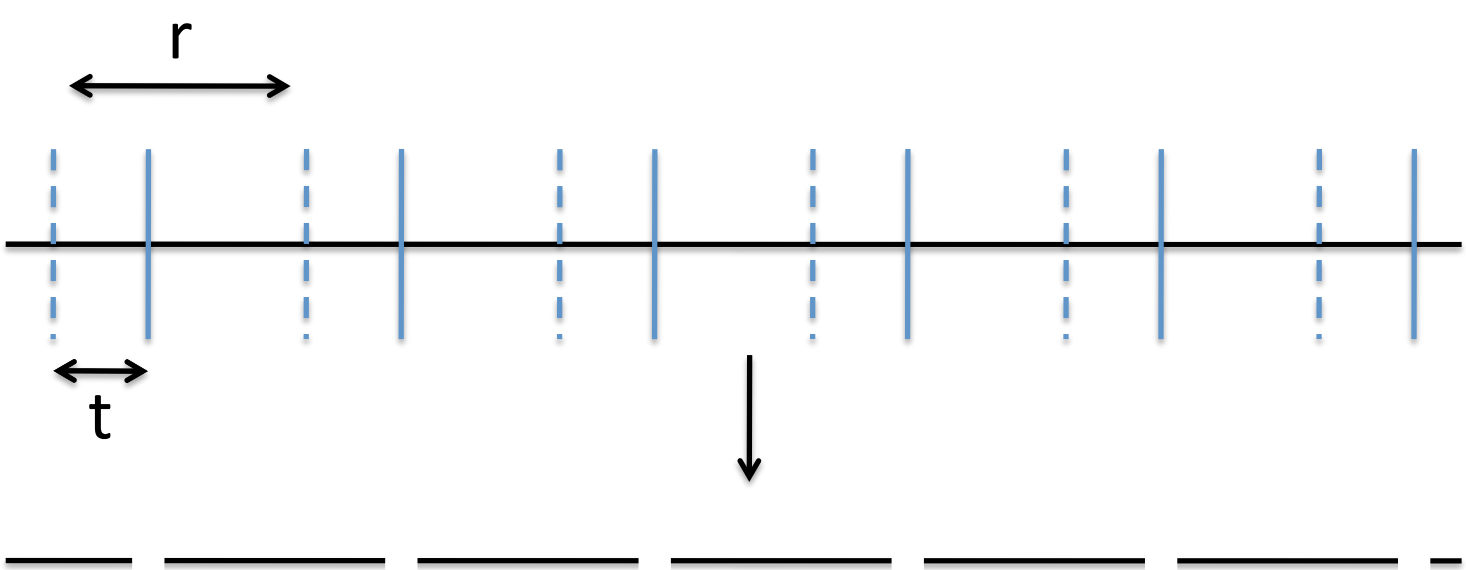
Random partitions

- Metric $M = (X,D)$
- Randomly partition X into $C_1, \ldots, C_k$
- $\mathrm{diam}(C_i) \leq r$
- $\Pr[C(x) \neq C(y)] \leq \beta \cdot D(x,y)/r$



Goal: small β (modulus of decomposability)

Example: Partitioning the reals



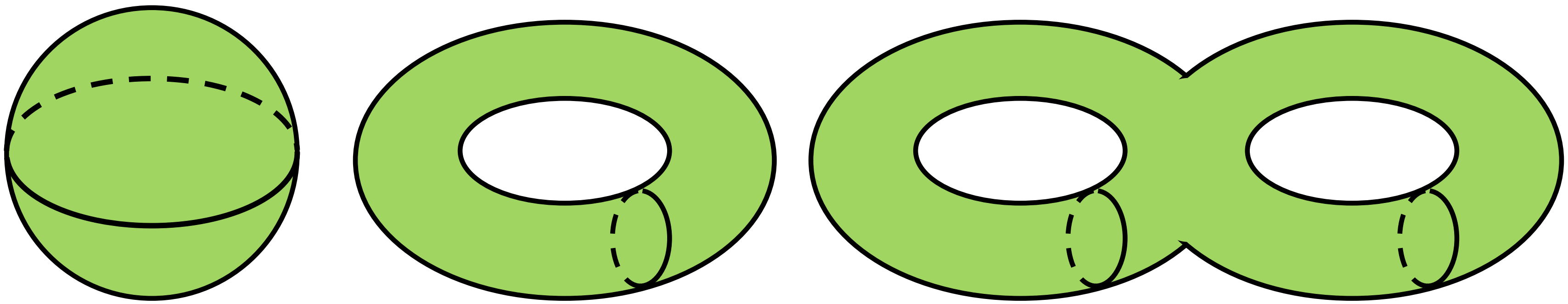
Pick t in [0,r) uniformly at random
Shift intervals by t

$\beta=1$

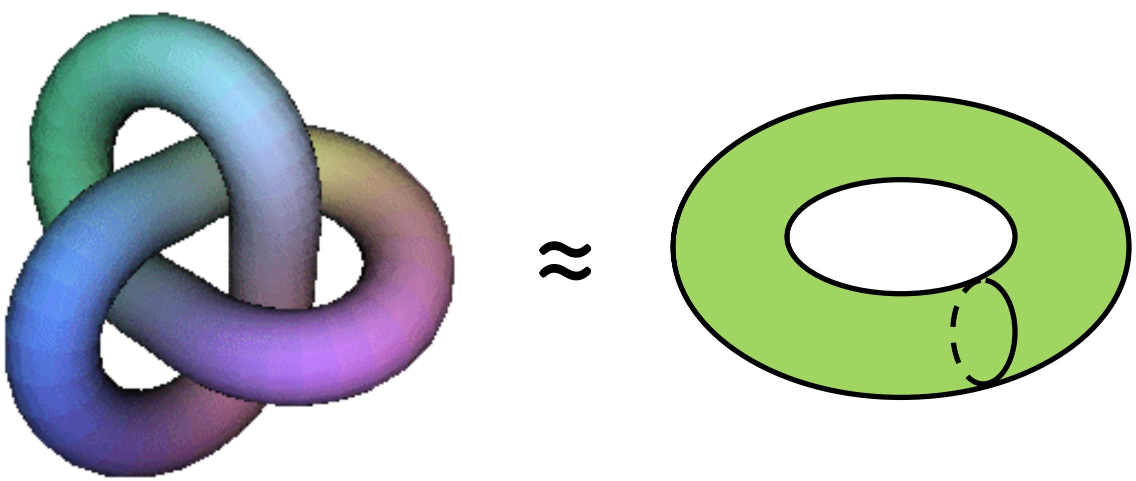
Random partitions

- General graphs:
  $\beta = O(\log n)$ [Bartal'95]
- Planar graphs:
  $\beta = O(1)$ [Klein,Plotkin,Rao'93]
- K_t-free graphs:
  $\beta = O(t^2)$ [KPR'93], [Fakcharoenphol,Talwar'03]
- Genus-g graphs:
  $\beta = O(g)$ [KPR'93], [FT'03]
  $\beta = O(\log g)$ [Lee,S'09]

Orientable surfaces

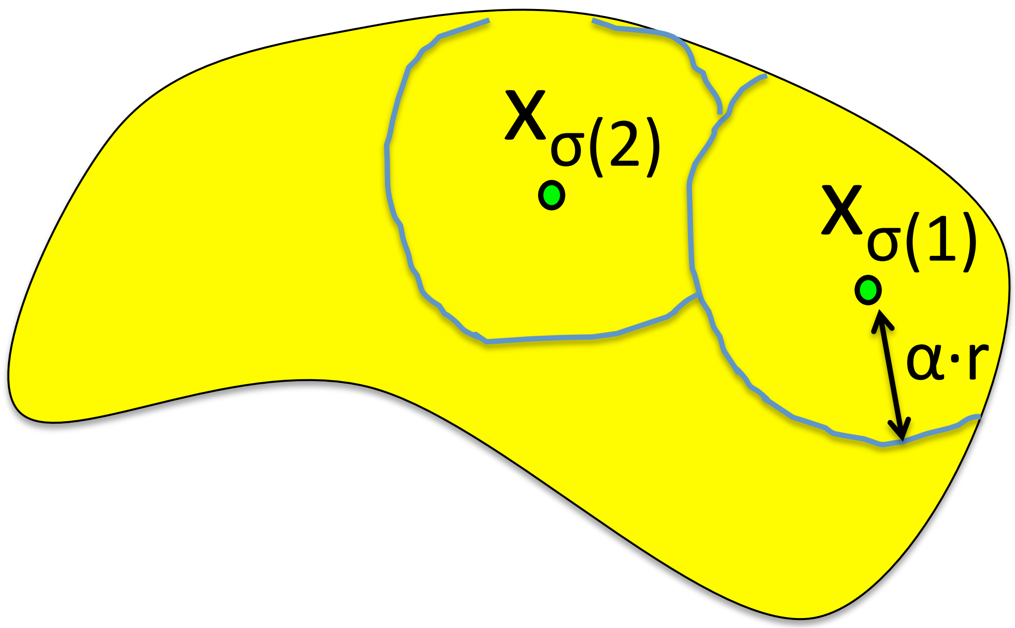


genus 0 genus 1 genus 2



Calinescu-Karloff-Rabani partitions

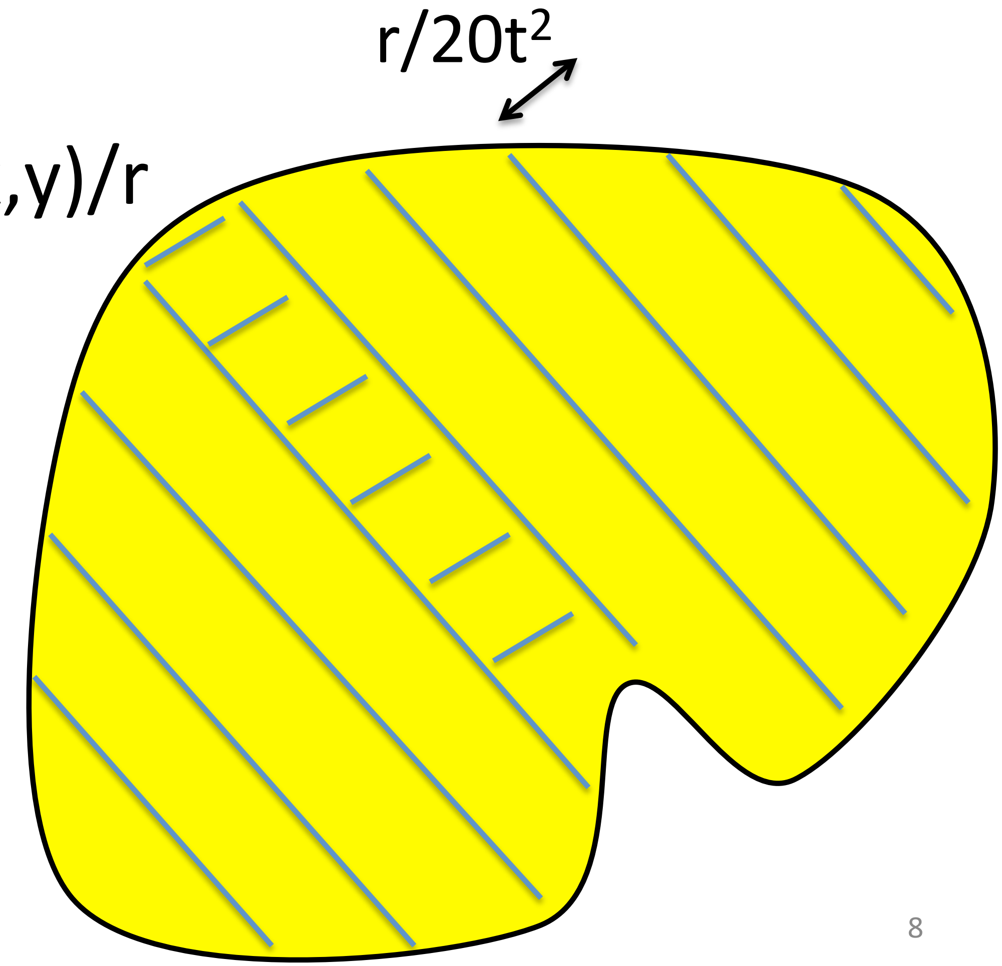
- Let $X=\{x_1,\ldots,x_n\}$
- Pick random σ in S_n
- Pick random α in [½,1)



Klein-Plotkin-Rao partitions

- Graph that excludes K_t as a minor
- Repeat t times
- $\Pr[C(x) \neq C(y)] \leq O(t^2) \cdot D(x,y)/r$

  [Fakcharoenphol,Talwar'03]

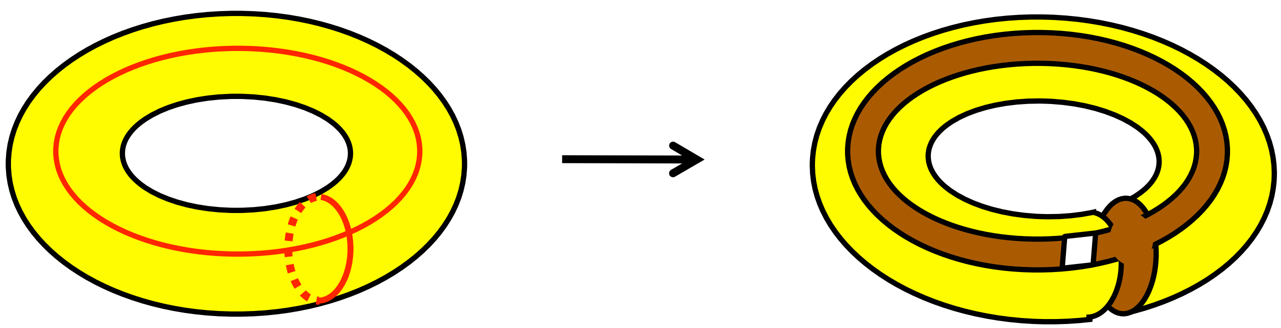


Genus

- Genus-t^2 graphs exclude K_t
- So, $\beta = O(g)$
- To get $O(\log g)$ we need a non-iterative approach
- We will remove all handles at once!
- Use homotopy generators [deVerdiere,Lazarus'02], [Erickson,Whittlesey'05]

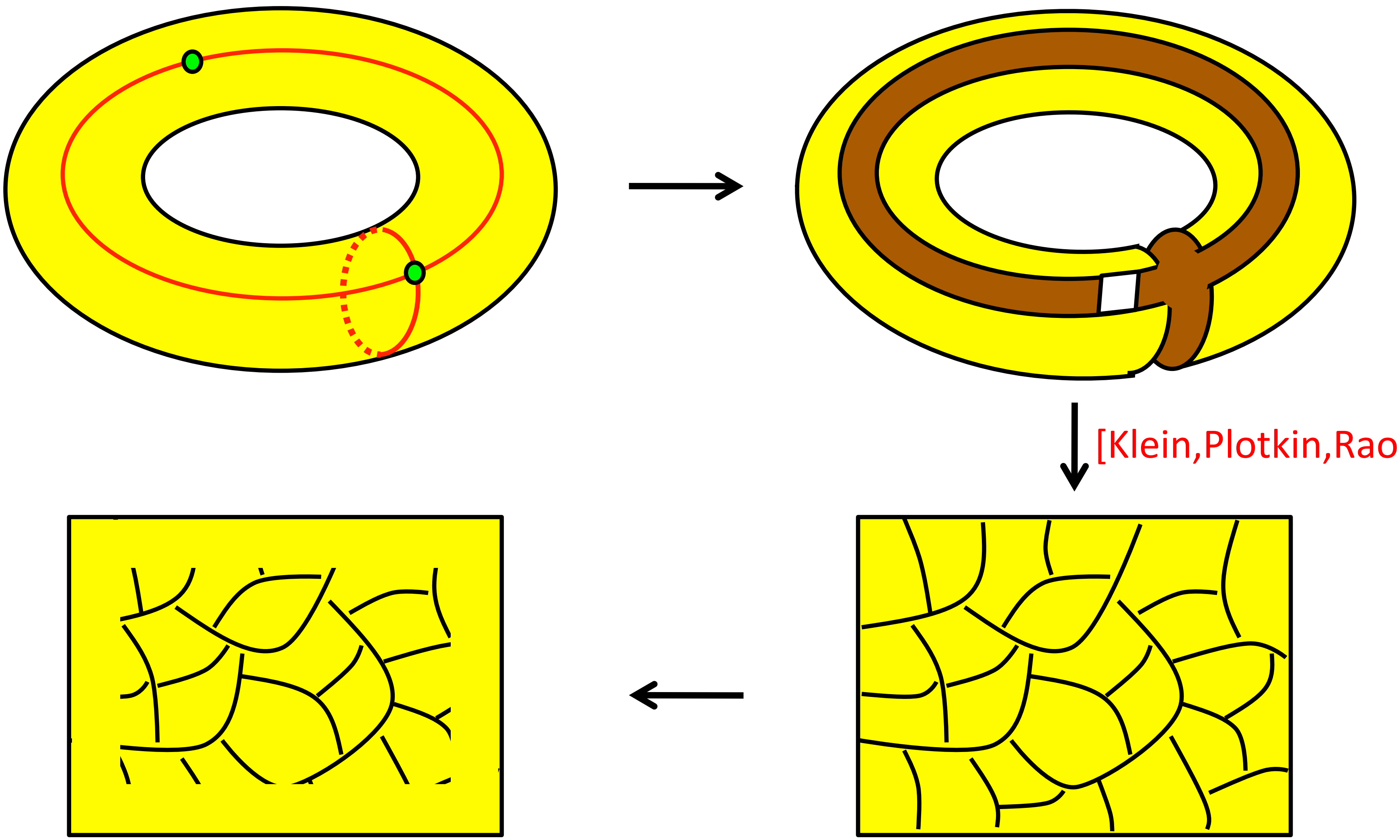
Homotopy generators

- Greedy system of loops [Erickson,Whittlesey'05]
  - Set H of cycles s.t. G\H is planar



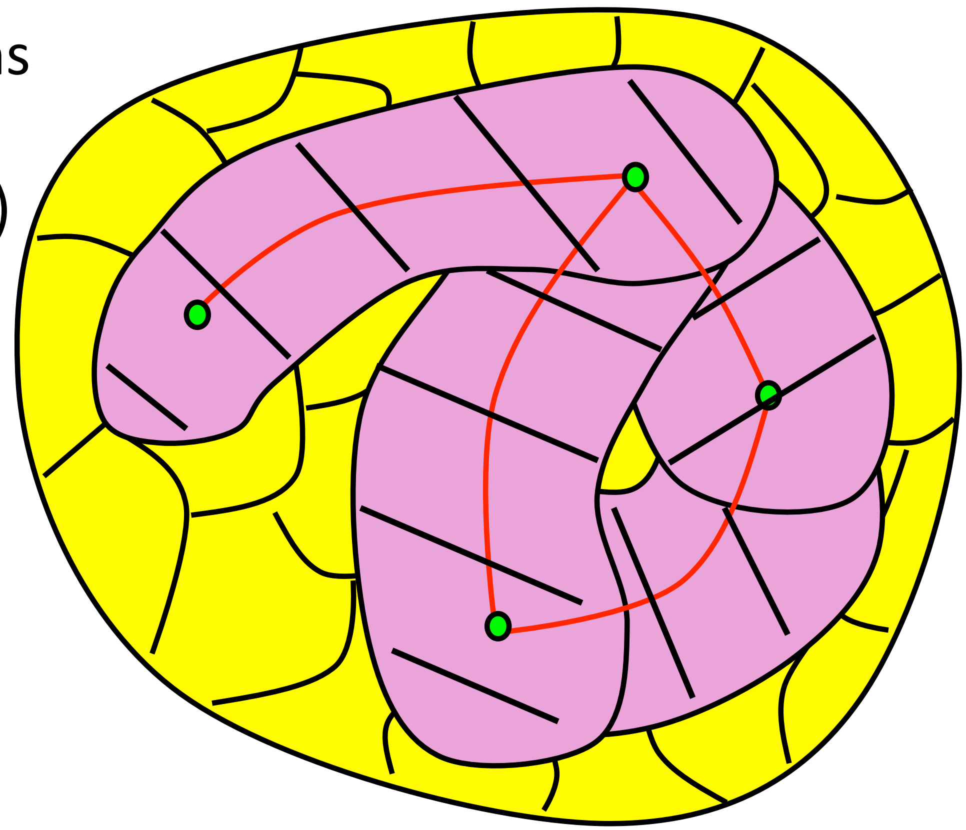
Fact: H consists of O(g) shortest paths

Partitioning surfaces



Partitioning surfaces

- Pick random permutation of paths
- Pr[C(x)≠C(y)]≤O(log g)·D(x,y)/r
  (by [Calinescu,Karloff,Rabani'01])



Implications

- O(log g)-approx for uniform Sparsest-cut [KPR'93]
- O(((log g)·(log n))$^{1/2}$)-approx for Non-uniform Sparsest-cut [Krauthgamer,Lee,Mendel,Naor'04]
- O(log g)-approx for treewidth [Feige,Hajiaghayi,Lee'05]
- k-th Laplacian eigenvalue: O(kg/n)·(log g)^2 [Kelner,Lee,Price,Teng'09]
- O(log g)-approx for 0-extension [Lee,Naor'04], [Calinescu,Karloff,Rabani'01]
- Similar improvements for Lipschitz extensions [Lee,Naor'04]
- Similar improvements for Minimum Crossing Number [Even,Guha,Schieber'02]

Open question

- Graphs that exclude K_t

  $Pr[C(x) \neq C(y)] \leq O(\log t) \cdot r / D(x,y)$?