# On the geometry of graphs with a forbidden minor

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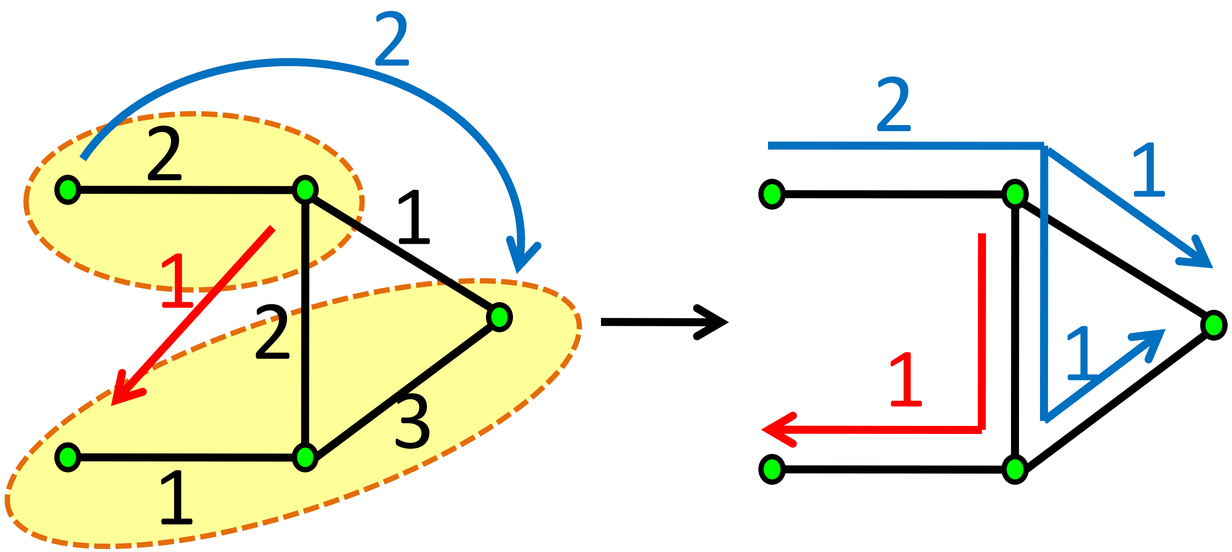
# Multi-commodity flows

**Instance:**

- G=(V,E)
- cap : E⟶R
- dem : V×V⟶R




**max-flow** = max concurrent flow

**sparsity of a cut** S = (capacity in S) / (demand crossing S)

**max-flow** ≤ **sparsest-cut**

# Approximating the Sparsest-Cut

- O(log n)-approximation [Linial,London,Rabinovich'95], [Leighton,Rao'88]
- O($\log^{1/2}$ n loglog n)-approximation [Arora,Lee,Naor'05], [Arora,Rao,Vazirani'04]
- 1.001-hard [Ambuhl,Mastrolilli,Svensson'07]
- ω(1)-hard assuming Unique Games [Khot, Vishnoi '05], [Chawla,Krauthgamer,Kumar,Rabani,Sivakumar '05]

# Sparsest-Cut and $L_1$ embeddings

**gap**(G) = $\max_{cap,dem}$ **sparsest-cut** / **max-flow**

$c_1$(G) = inf{c : G embeds into $L_1$ with distortion c}

**Theorem** [Linial,London,Rabinovich'95] [Aumann,Rabani'98]
For every graph G, **gap**(G)=$c_1$(G)

**Conjecture** [Gupta,Newman,Rabinovich,Sinclair'99] Every nontrivial minor-closed graph family embeds into $L_1$ with distortion O(1)

# Progress on the GNRS conjecture

- Series-parallel graphs [Gupta,Newman,Rabinovich,Sinclair ’99]
- O(1)-outerplanar graphs [Chekuri,Gupta,Newman,Rabinovich,Sinclair ‘03]
- $W_4$-free graphs [Chakrabarti,Jaffe,Lee,Vincent’08]
- If true for planar graphs, then also true for O(1)-genus graphs [Indyk, S ‘07]
- 2-ε lower bound for planar graphs [Lee,Raghaventra’07]

# Our results

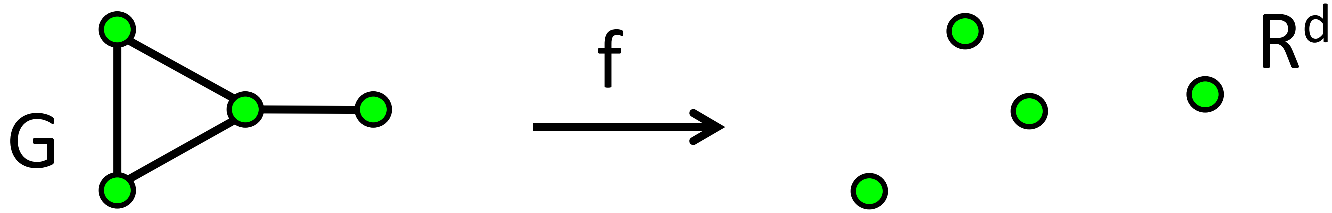
**Theorem** [Lee, S'09] The GNRS is true for O(1)-pathwidth graphs

**Theorem** [Lee, S'09] The GNRS is true iff

- planar graphs embed into $L_1$ with distortion O(1), and
- O(1)-distortion embeddability into $L_1$ is closed under O(1)-clique-sums

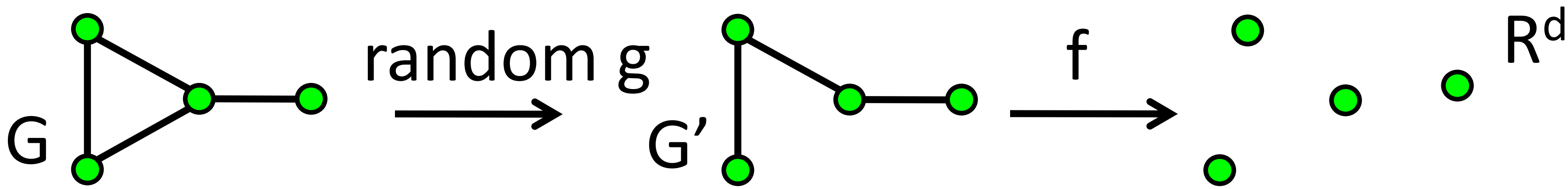
# The main tool: Stochastic embeddings

- Deterministic embeddings into $L_1$




$$|f(x)-f(y)|_1 \le d(x,y) \le \boldsymbol{\beta} \cdot |f(x)-f(y)|_1$$

- Stochastic embedding approach

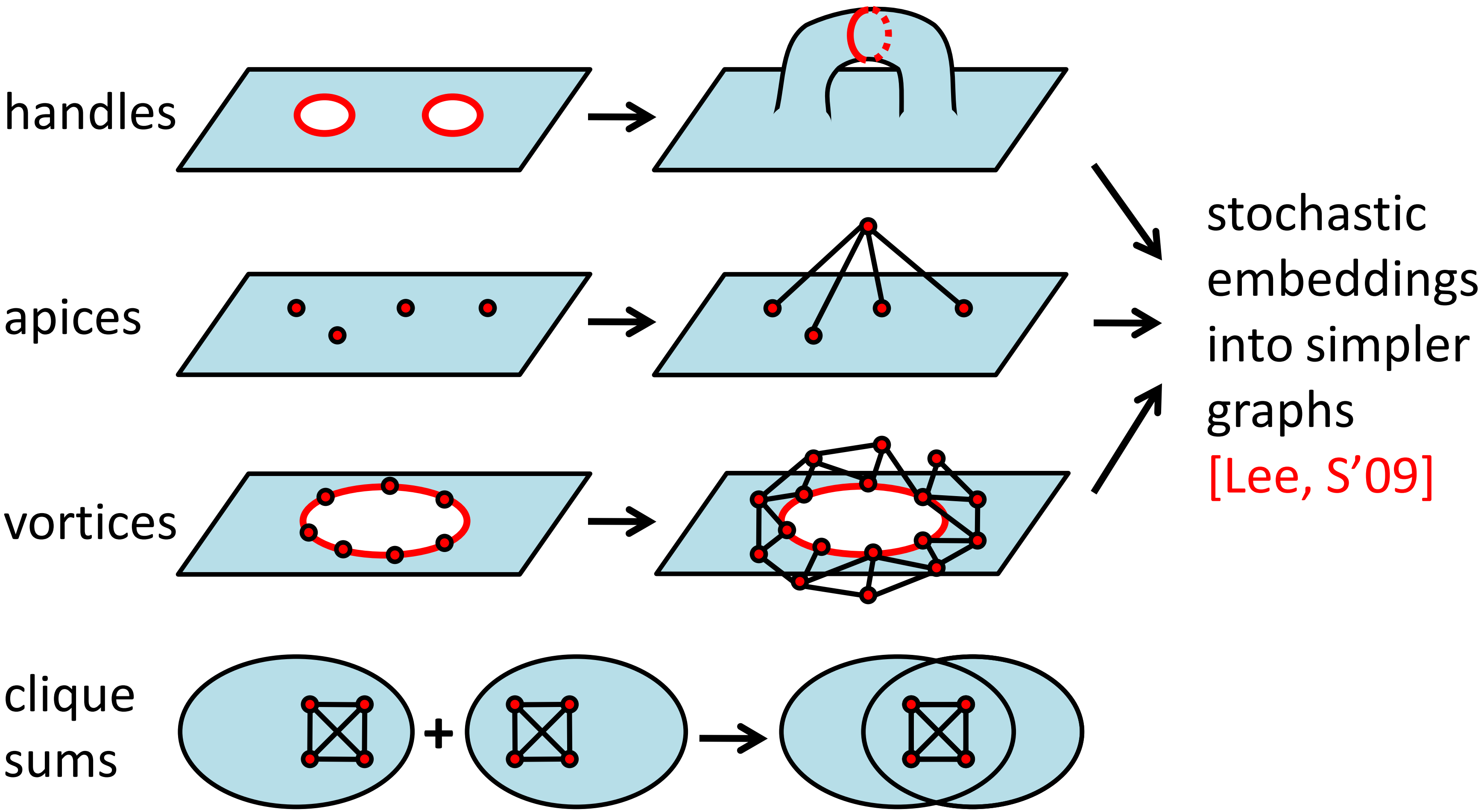



$$\Pr[d'(g(x),g(y)) \ge d(x,y)] = 1$$

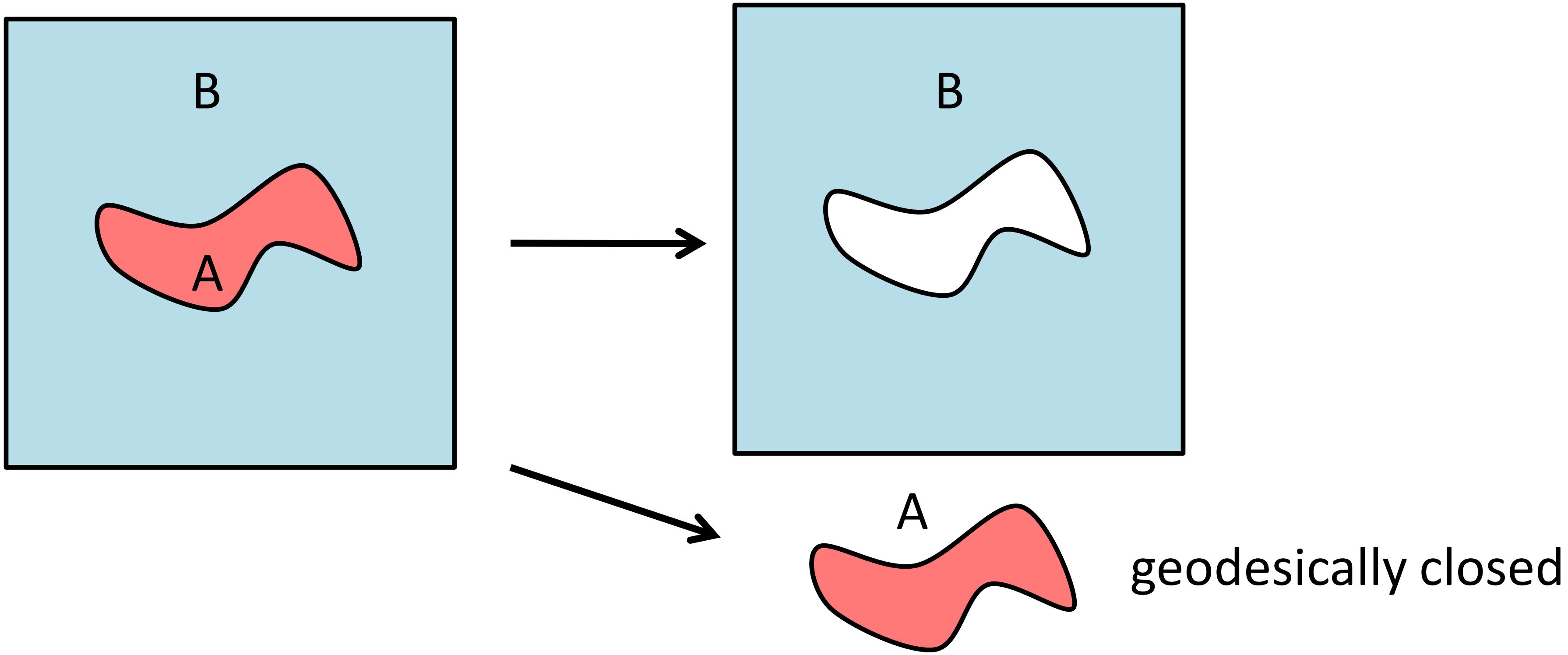
$$E[d'(g(x),g(y))] \le \boldsymbol{\alpha} \cdot d(x,y)$$

# The Graph Minor Theorem

How to construct any non-trivial minor-closed graph family
[Robertson, Seymour ’99]



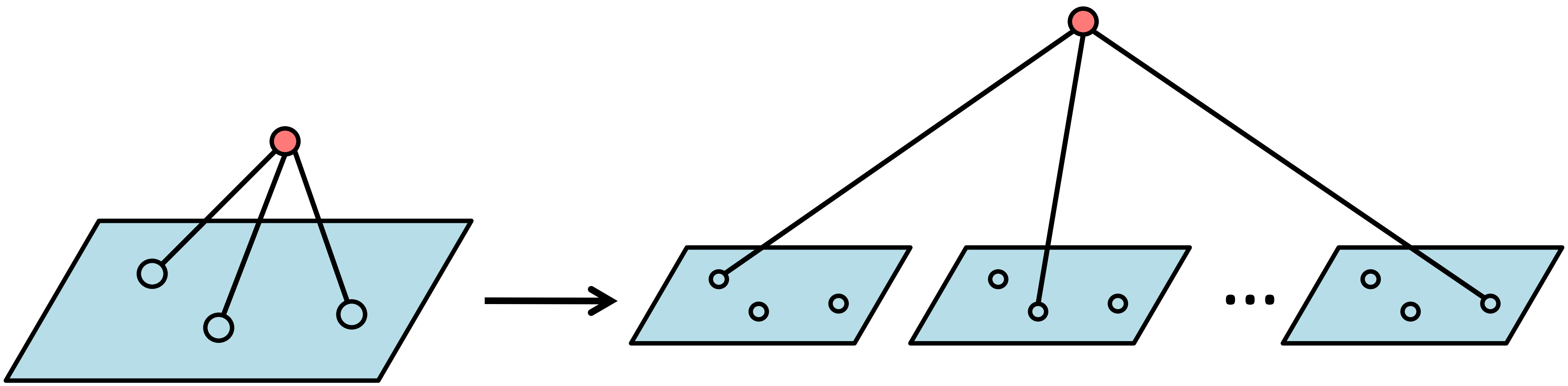
# The peeling lemma



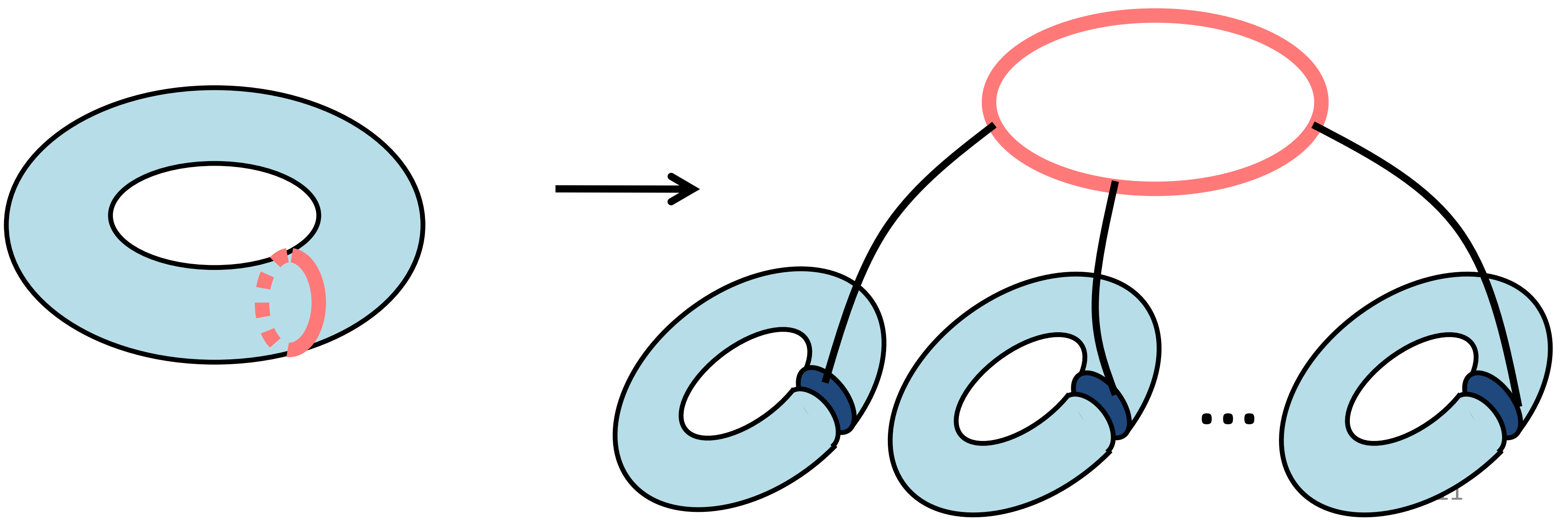

**Peeling Lemma** [Lee, S ’09]

$A \cup B$ stochastically O(1)-embeds into 1-sums of A with B

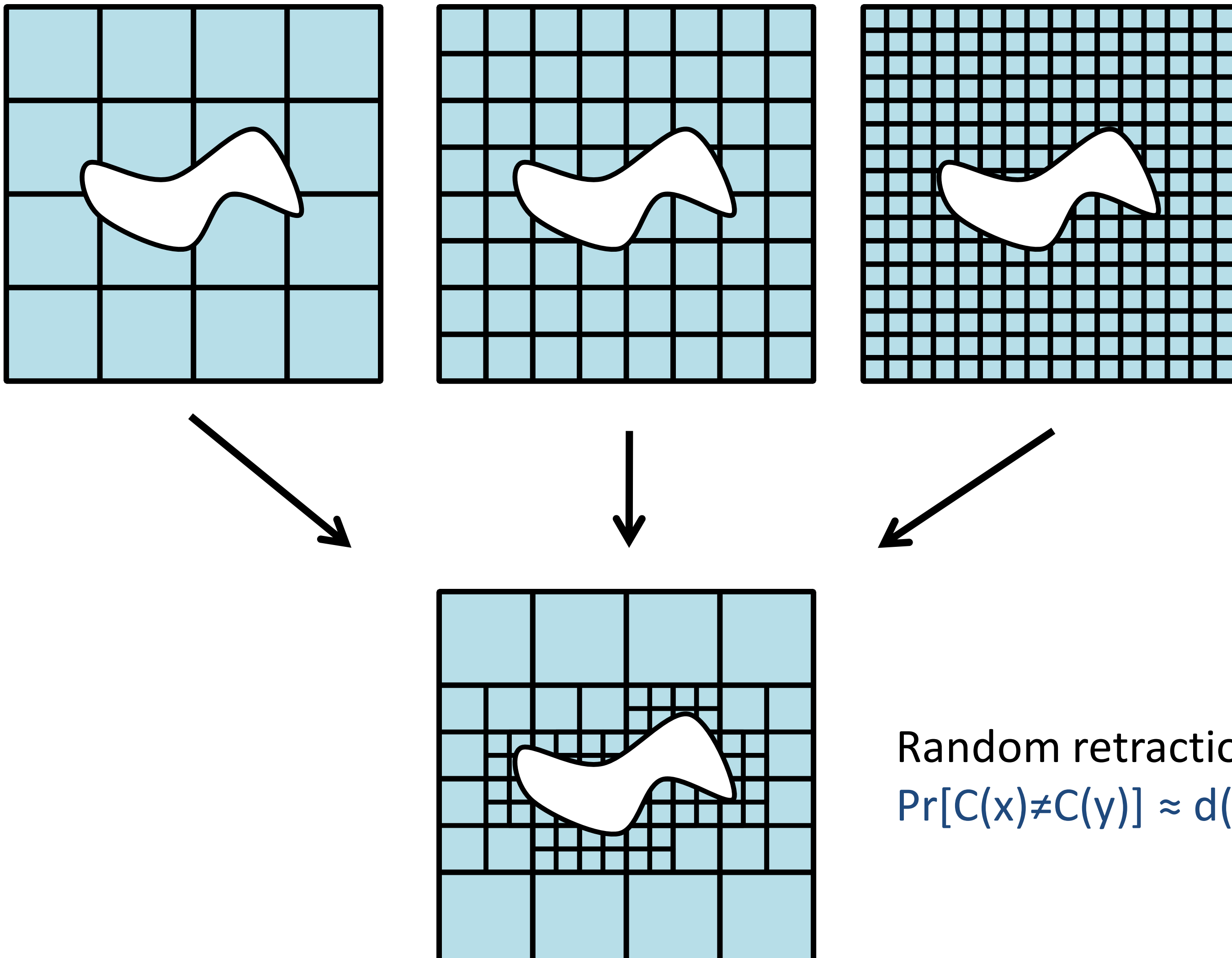
# The peeling lemma: removing apices



# The peeling lemma: removing handles



# The peeling lemma (proof)

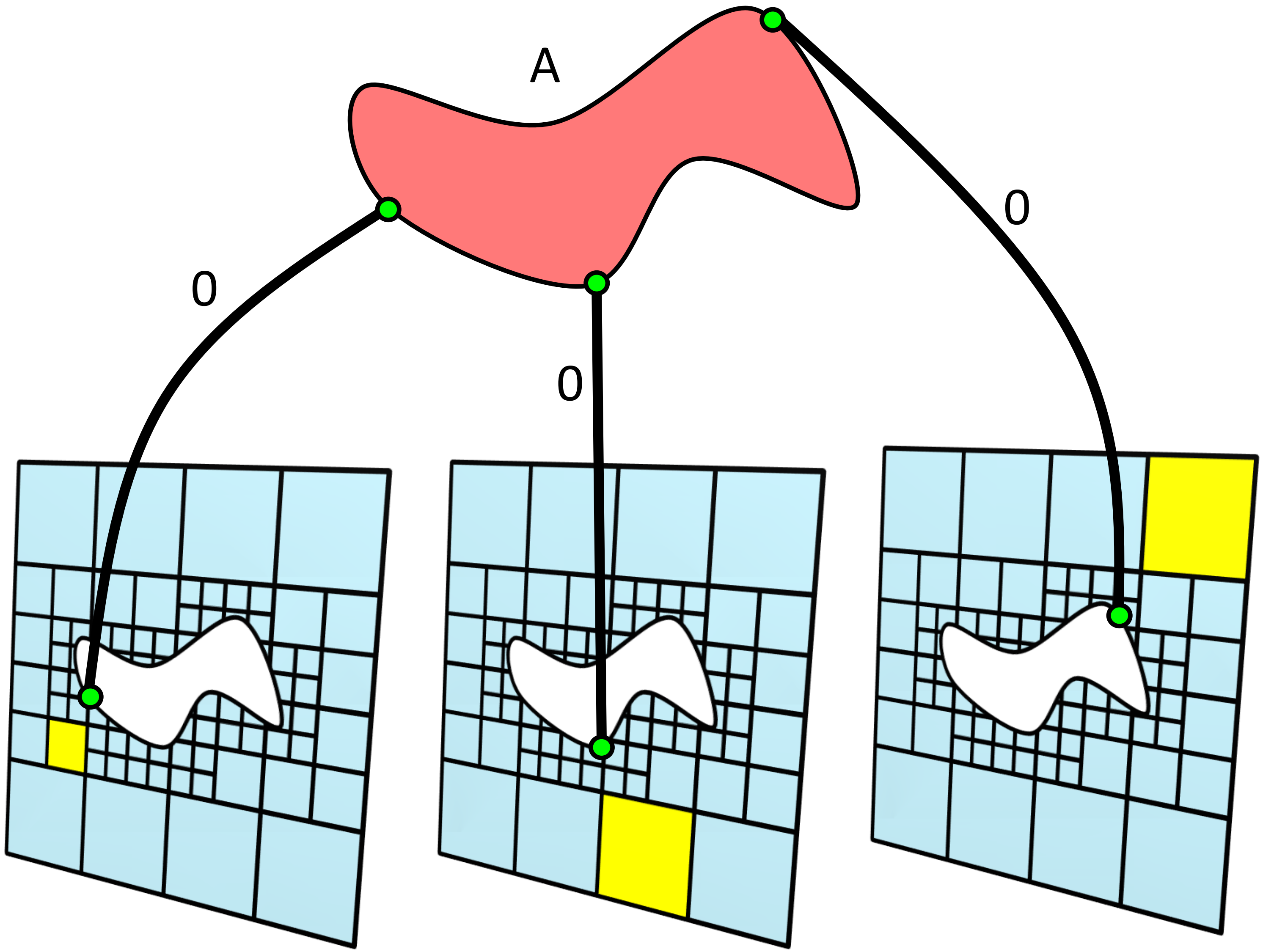


Random decomposition for every scale [Klein,Plotkin,Rao'93]

At scale r:
Pr[C(x)≠C(y)] ≈ d(x,y)/r

Random retraction [Lee, Naor]
Pr[C(x)≠C(y)] ≈ d(x,y) / d({x,y}, A)

# The peeling lemma (proof)



# Further applications

**Theorem** [Okamura,Seymour ‘81] For any multi-flow instance with terminals on a face of a planar graph, **max-flow**=**min-cut**

**Theorem** [Lee,S’09] For any multi-flow instance with terminals on **O(1)** faces of a **O(1)**-genus graph, **max-flow**=**O(1)**·**min-cut**

# Open questions

- **What are the actual constants?**
  - **Recent progress:** $\tilde{O}(\log^{1/2} g)$-approximation for uniform Sparsest-Cut on genus-g graphs [Lee,S]

    Improves $O(g^2)$-approximation [Fakcharoenphol,Talwar'03], [Klein,Plotkin,Rao'93]

- **Embedding into $L_1$:**
  - Embedding planar graphs with distortion O(1)?