Randomly Removing g Handles at Once

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ABSTRACT

It was shown in [11] that any orientable graph of genus g can be probabilistically embedded into a graph of genus $g-1$ with constant distortion. Removing handles one by one gives an embedding into a distribution over planar graphs with distortion $2^{O(g)}$. By removing all g handles at once, we present a probabilistic embedding with distortion $O(g^2)$ for both orientable and non-orientable graphs. Our result is obtained by showing that the minimum-cut graph of [6] has low dilation, and then randomly cutting this graph out of the surface using the Peeling Lemma from [13].

Categories and Subject Descriptors

F.2 [**Analysis of Algorithms and Problem Complexity**]: General

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Embeddings, Probablistic Approximation, Bounded Genus Graphs, Planar Graphs

1. INTRODUCTION

Planar graphs constitute an important class of combinatorial structures, since they can often be used to model a wide variety of natural objects. At the same time, they have properties that give rise to improved algorithmic solutions for numerous graph problems, if one restricts the set of possible inputs to planar graphs (see e.g. [1, 3]).

One natural generalization of planarity uses the genus of a graph. Informally, a graph has genus g, for some $g \geq 0$, if it can be drawn without any crossings on the surface of a sphere with g additional handles (see Section 1.1). For example, a planar graph has genus 0, and a graph that can be drawn on a torus has genus 1.

In a way, the genus of a graph quantifies how far a graph is from being planar. Because of their similarities to planar graphs, graphs of small genus usually exhibit nice algorithmic properties. More precisely, algorithms for planar graphs can usually be extended to graphs of bounded genus, with a small loss in efficiency (e.g. [4]), or in the quality of the solution. Unfortunately, some such extensions are complicated and based on ad-hoc techniques.

Inspired by Bartal's probabilistic approximation of general metrics by trees [2], Sidiropoulos and Indyk showed that every metric on a graph of genus g can be probabilistically approximated by a planar graph metric with distortion at most exponential in g [11]. (See Section 1.1 for a formal definition of probabilistic embeddings, a randomized mapping between spaces preserving distances in expectation). Since the distortion fundamentally affects the quality of the reduction, it is desirable to make this quantity as small as possible. In the present paper, we show that the dependence of the distortion on the genus can be made significantly smaller: $O(g^2)$ for graphs of orientable or non-orientable genus g. This requires a fundamental change over the approach of [11] which removes one handle at a time from the graph.

Removing all the handles at once. Since (randomly) removing handles one at a time incurs an exponential loss in distortion, we look for a way to remove all the handles at once.

Our starting point is the *minimum-length cut graph* of Erickson and Har-Peled [6]. Given a graph G the minimum-length cut graph is (roughly speaking) a minimum-length subgraph H of G such that $G \setminus H$ is planar. In Section 2.2, we show that this H is nearly *geodesically closed* in that $d_H(u,v) \approx d_G(u,v)$ for all $u, v \in V(H)$, where d_H and d_G are the shortest-path metrics on H and G, respectively. Simply removing H from G could result in unbounded distortion for some pairs of vertices of G. The geodesic-closure property suggests that if we could *randomly shift* H, then the distortion of all pairs of vertices in G would be fine in expectation.

We use the *Peeling Lemma* of [13] to perform the random shifting (Section 2.1). The Peeling Lemma allows one to randomly embed G into a graph consisting of copies of $G \setminus H$ hanging off an isomorphic copy of H, while keeping the expected distortion of pairs of vertices in G small. The lemma requires an appropriate random partition of the

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shortest-path metric on G. Such a procedure is provided by the fundamental result of Klein, Plotkin, and Rao [12] for partitioning graphs excluding a fixed minor.

This completes the proof, except that H itself might not be planar. However, H does have small Euler characteristic[1] [6] and so admits a probabilistic embedding into a distribution over trees [9, 7]. In Section 2.3, we combine these ingredients to provide a probabilistic embedding with distortion $O(g^2)$.

In Section 3, we show that any such probabilistic embedding incurs at least $\Omega(\log g)$ distortion. (A lower bound of $\Omega(\log g/\log\log g)$ was given in [11].) This still leaves an exponential gap between our upper in lower bounds. We study the limitations of our particular techniques and show an $\Omega(g)$ lower bound for a restricted class of approaches.

1.1 Preliminaries

Throughout the paper, we consider graphs $G = (V, E)$ with a non-negative length function $\mathsf{len} : E \to \mathbb{R}$. We refer to these as *metric graphs*. For pairs of vertices $u, v \in V$, we denote the length of the shortest path between u and v in G, with the lengths of edges given by len, by $d_G(u, v)$.

Graphs on surfaces.

Let us recall some notions from topological graph theory (an in-depth exposition can be found in [15]). A *surface* is a compact connected 2-dimensional manifold, without boundary. For a graph G we can define a one-dimensional simplicial complex C associated with G as follows: The 0-cells of C are the vertices of G, and for each edge $\{u, v\}$ of G, there is a 1-cell in C connecting u and v. An *embedding* of G on a surface S is a continuous injection $f : C \to V$. The *orientable genus* of a graph G is the smallest integer $g \geq 0$ such that C can be embedded into a sphere with g handles. The *non-orientable genus* of G is the smallest ineger $k \geq 0$ such that G can be embedded into a sphere with k disjoint caps replaced by copies of the projective plane. Note that a graph of genus 0 is a planar graph.

Metric embeddings.

We use the notion of *stochastic embeddings* introduced in [2]. We say that a graph H *dominates* a graph G, if $V(G) \subseteq V(H)$, and for any $u, v \in V(G)$, $d_G(u, v) \leq d_H(u, v)$. Let $\mathcal{G}$ be a family of graphs, and let $\alpha \geq 1$. A *stochastic α-embedding* of a graph G into $\mathcal{G}$ is a probability distribution over graphs $H \in \mathcal{G}$ that dominate G, such that for any $u, v \in V(G)$,

$$\mathbf{E}[d_H(u, v)] \leq \alpha \cdot d_G(u, v).$$

For graph families $\mathcal{F}$, $\mathcal{G}$ we write $\mathcal{F} \overset{\alpha}{\rightsquigarrow} \mathcal{G}$ if every graph $G \in \mathcal{F}$ admits a stochastic α-embedding into $\mathcal{G}$. A detailed exposition of results on metric embeddings can be found in [14], and [10].

2. RANDOM PLANARIZATION

We now show that every metric graph of orientable or non-orientable genus g embeds into a distribution over planar graph metrics with distortion at most $O(g^2)$.

[1]The Euler characteristic we refer to exclusively in this paper is the value $|E| - |V| + 1$.

2.1 The peeling lemma

In this section, we review the Peeling Lemma from [13]. Let $G = (V, E)$ be a metric graph, and consider any subset $A \subseteq V$. Let $G[A]$ denote the subgraph of G induced by A, and let $d_{G[A]}$ denote the induced shortest-path metric on A. The *dilation of A in G* is

$$\mathsf{dil}_G(A) = \max_{x \neq y \in A} \frac{d_{G[A]}(x, y)}{d_G(x, y)}.$$

Since $d_{G[A]}(x, y) \geq d_G(x, y)$ for all $x, y \in A$, $\mathsf{dil}_G(A) \geq 1$.

We now recall the following definition.

Definition 1 (Lipschitz random partition). *For a partition P of a set X, we write $P : X \to 2^X$ to denote the map which sends x to the set $P(x) \in P$ which contains x. A random partition P of a finite metric space X is Δ-bounded if*

$$\Pr[\forall C \in P, \mathrm{diam}(C) \leq \Delta] = 1.$$

A Δ-bounded random partition P is β-Lipschitz if, for every $x, y \in X$,

$$\Pr[P(x) \neq P(y)] \leq \beta \frac{d(x, y)}{\Delta}.$$

For a metric space (X, d), we write $\beta_{(X,d)}$ for the infimal β such that X admits a Δ-bounded β-Lipschitz random partition for every $\Delta > 0$, and we refer to $\beta_{(X,d)}$ as the *decomposability modulus of X*.

The results of Rao [16] and Klein, Plotkin, and Rao [12] yield the following for the special case of bounded-genus metrics. (The stated quantitative dependence is due to [8].)

Theorem 1 (KPR Decomposition). *If $G = (V, E)$ is a metric graph of orientable or non-orientable genus $g \geq 0$, then $\beta_{(V,d_G)} = O(g + 1)$.*

The dilation and modulus is used in the statement of the Peeling Lemma. We use $G \overset{D}{\rightsquigarrow} H$ to denote the fact that G admits a stochastic D-embedding into the family $\{H\}$ (consisting only of the single graph H).

Lemma 1 (Peeling Lemma, [13]). *Let $G = (V, E)$ be a metric graph, and $A \subseteq V$ an arbitrary subset of vertices. Let $G' = (V, E')$ be the metric graph with $E' = E \setminus E(G[A])$, and let $\beta = \beta_{(V,d_{G'})}$ be the corresponding modulus of decomposability. Then $G \overset{D}{\rightsquigarrow} H$, where $D = O(\beta \cdot \mathsf{dil}_G(A))$, and H is a 1-sum of isometric copies of the metric graphs $G[A]$ and $\left\{G[V \setminus A \cup \{a\}]\right\}_{a \in A}$. Furthermore, the embedding always has distortion at most $\mathsf{dil}_G(A)$ for pairs $x, y \in A$.*

2.2 Low-dilation planarizing sets

In light of the Peeling Lemma, given a graph of bounded genus, we would like to find a low-dilation set A whose removal leaves behind planar components. In section 2.3, we will deal with the fact that the $G[A]$ might not be planar. In everything that follows, $\mathbb{S}$ will denote some compact surface of bounded (orientable or non-orientable) genus.

Definition 2 (Cut graph [6]). *Let G be a graph embedded in $\mathbb{S}$. Then, a subgraph H of G is called a* cut graph *if cutting $\mathbb{S}$ along the image of H results in a space homeomorphic to the disk.*

Definition 3 (One-sided walk). *Let D be the disk obtained by a cut graph. Every edge of H appears twice in the boundary of D. Let x and y be two unique vertices on the boundary of D. Let R and R' be the paths bounding D between x and y. An x-to-y walk X is called* one-sided *if for every edge e of X, e is in R and e's copy, e' is in R'.*

Lemma 2. *For any cut graph H and any two vertices x, y of H, there is a one-sided walk from x to y in H.*

Proof. Let R and R' be the x-to-y boundaries of the disk upon cutting the surface along H. Let C be the set of edges both of whose copies are in R. Let A be the subgraph of H containing all the edges in R. A contains an x-to-y path. Let B be the subgraph of H containing all the edges in $R \setminus C$. We prove B contains an x-to-y path.

Let e be any edge in C and let e' be its copy. Removing e and e' from R is equivalent to glueing the disk along e and e', creating a punctured surface. Since edges of R' are never glued together, they will remain on the boundary of a common puncture. Let S be the boundary of this puncture after glueing all the edges of C together. Since R' is an x-to-y walk, $S \setminus R'$ is an x-to-y walk. All the edges in $S \setminus R'$ are in R and their copies are in R' by construction. Therefore $S \setminus R'$ is a one-sided walk. □

First, we bound the dilation of certain cut graphs.

Lemma 3. *Let G be a graph embedded in $\mathbb{S}$. If H is a cut graph of G of minimum total length and h is the number of vertices of degree at least 3 in H, then*

$$\mathsf{dil}_G(V(H)) \le h + 2.$$

Proof. Assume for the sake of contradiction that there exist $x, y \in V(H)$, with

$$d_H(x, y) > (h + 2)\, d_G(x, y),$$

and pick x, y so that $d_G(x, y)$ is minimized among such pairs. Let Q be a shortest path between x and y in G. Observe that by the choice of x, y, the path Q intersects H only at x and y. Let D be the disk obtained after cutting $\mathbb{S}$ along H. Since $Q \cap H = \{x, y\}$, it follows that the interior of Q is contained in the interior of D (Figure 1(a)).

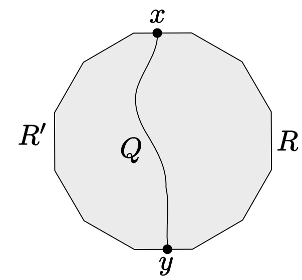
Let R, R' be the paths in D between the end-points of Q, obtained by traversing clockwise the boundary of D starting from x, and y respectively (Figure 1(a)). Let S be the one-sided walk that is contained in R, as guaranteed by Lemma 2. Let J be any simple x-to-y path contained in S. We have

$$\mathsf{len}(J) \ge d_H(x, y) > (h + 2)\, d_G(x, y) = (h + 2)\mathsf{len}(Q) \quad (1)$$

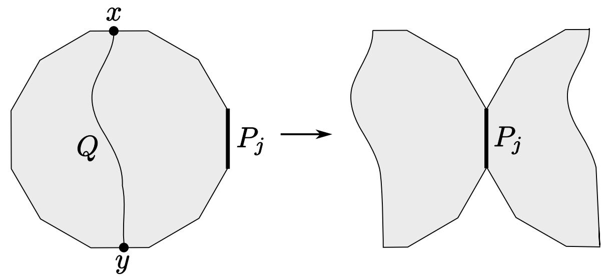
Let U be the set of vertices of H of degree at least 3. Since J is a simple path, it visits each vertex in U at most once. It follows that the path J consists of at most $k \le h + 1$ paths $P_1, \ldots, P_k$ in H. Since J is a one-sided path, exactly one copy of P_i appears in R, for each $i \in [k]$. Let $j = \operatorname{argmax}_{i \in [k]} \mathsf{len}(P_i)$. By (1), we have

$$\mathsf{len}(P_j) \ge \frac{\mathsf{len}(J)}{h + 1} > \frac{h + 2}{h + 1}\mathsf{len}(Q) > \mathsf{len}(Q). \quad (2)$$

Since P_j has exactly one copy in each of R, and R', it follows that by cutting D along Q and gluing back along P_j we end up with a space homeomorphic to a disk (Figure 1(b)). Therefore, the graph H', obtained from H by removing P_j and by adding Q, is a cut graph. By (2), H' has smaller total length than H, contradicting the minimality of H, and concluding the proof. □



(a) The disk $\mathbb{S} \setminus \bar{H}$ with R, R' on its boundary.



(b) Cutting along Q and glueing along P_j.

Figure 1: Building a smaller cut graph.

Now we state a result of [6].

Theorem 2 (Minimum-length cut graph, [6]). *Let G be a graph embedded on a surface $\mathbb{S}$ of orientable or non-orientable genus $g \ge 1$. Then every minimal-length cut graph H in G has at most $4g - 2$ vertices of degree at least 3. Moreover, H is the subdivision of some graph with at most $4g - 2$ vertices and at most $6g - 3$ edges.*

2.3 Applying the Peeling Lemma

Given Lemma 3 and Theorem 2, we are in position to applying the Peeling Lemma, except that the cut graph H of Theorem 2 might not itself be planar.

Theorem 3. *Let G be any metric graph of orientable or non-orientable genus $g \ge 1$. Then G admits a stochastic $O(g^2)$-embedding into a distribution over planar graph metrics.*

Proof. Let G be embedded into a surface $\mathbb{S}$ of orientable or non-orientable genus $g \ge 1$. Let C be the cut graph given by Theorem 2, which has at most $4g - 2$ vertices of degree at least 3. Applying Lemma 3, we have

$$\mathsf{dil}_G(V(C)) \le 4g. \quad (3)$$

Since cutting $\mathbb{S}$ along C gives a space homeomorphic to a disk, it follows that $G \setminus C$ is planar. $G \setminus C$ might be disconnected, but this does not affect the argument. See Figure 2(a).

Assume, without loss of generality, that the minimum distance in G is 1. (Since G is finite, the distances can always be rescaled to satisfy this constraint.) Let J be the graph obtained from G as follows. For every edge $\{u,v\} \in E(G)$, with $u \in V(C)$, and $v \notin V(C)$, we introduce a new vertex z, and we replace $\{u,v\}$ by a path u-z-v, with $d_J(z,v) = \frac{1}{2}$, and $d_J(u,z) = d_G(u,v) - \frac{1}{2}$. Let Z be the set of all these new vertices, and let $K = J[V(C) \cup Z]$, i.e. K is the subgraph obtained from C by adding all the new vertices in Z, and all the edges between Z and $V(C)$. Observe that for any $x, y \in V(G)$,

$$d_J(x,y) = d_G(x,y).$$

Therefore G is an isometric subgraph of J. For any $x, y \in V(K)$, let x', y' be the nearest neighbors of x and y in $V(C)$, respectively. Since the minimum distance in J is at least $\frac{1}{2}$, we have

$$\begin{aligned} d_K(x,y) &= d_K(x',y') + d_K(x,x') + d_K(y,y') \\ &\leq 1 + d_C(x',y') \\ &\leq 1 + 4g\, d_G(x',y') \text{ by Equation (3)} \\ &= 1 + 4g\, d_J(x',y') \\ &\leq 1 + 4g\,(d_J(x,y) + d_J(x,x') + d_J(y,y')) \\ &\leq 1 + 4g + 4g\, d_J(x,y) \\ &\leq (4g + 2(4g+1)) d_J(x,y) \\ &\leq 14g\, d_J(x,y) \end{aligned}$$

Therefore,

$$\mathsf{dil}_J(V(K)) \leq 14g. \tag{4}$$

Since G is a graph of genus $g > 0$, it follows that the modulus of decomposability of $J \setminus K$ is

$$\beta(J \setminus K) = \beta(G \setminus C) = O(g) \tag{5}$$

by Theorem 1.

Thus, by the Peeling Lemma and by (4) and (5), we obtain that J can be embedded into a distribution $\mathcal{F}$ over graphs obtained by 1-sums of K with copies of $\{J[V(J) \setminus V(K) \cup \{a\}]\}_{a \in V(K)}$, with distortion at most $O(\beta(J \setminus K) \cdot \mathsf{dil}_J(V(K))) = O(g^2)$. Observe that $J \setminus K = G \setminus C$, and thus $J \setminus K$ is a planar graph. Moreover, for any $a \in V(K)$, there is at most one edge between a and $J \setminus K$, and thus the graph $J[V(J) \setminus V(K) \cup \{a\}]$ is planar. In other words, any graph in the support of $\mathcal{F}$ is obtained by 1-sums between K and several planar graphs.

It remains to planarize K. We observe that for pairs $x, y \in K$, we have

$$d_{G[K]}(x,y) \leq \mathsf{dil}_G(V(K))\, d_G(x,y) = O(g)\, d_G(x,y),$$

by the final statement of the Peeling Lemma. We embed K into a random tree with distortion $O(\log g)$, yielding an embedding of G into planar graphs with total distortion at most $O(g^2)$ (in fact, pairs in K are stretched by only $O(g \log g)$ in expectation).

By Theorem 2, the graph C is the subdivision of a graph C' with at most $4g-2$ vertices and at most $6g-3$ edges. Recall that for a graph $\Gamma = (V,E)$, its Euler characteristic is defined to be $\chi(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$. Clearly, the Euler characteristic of a graph does not change by taking subdivisions, so we have

$$\chi(C) = \chi(C') < |E(C')| - |V(C')| + 1 < 6g. \tag{6}$$

Combining the stochastic embedding of arbitrary graphs into trees from [7], with the stochastic embedding of graphs with small Euler characteristic into trees from [9], it follows that every graph Γ embeds into a distribution over dominating trees with distortion at most $O(\log \chi(\Gamma))$. Therefore by (6) we obtain that C can be embedded into a distribution $\mathcal{D}$ over dominating trees with distortion $O(\log g)$. Let T be a random tree sampled from this distribution. Let T' be the graph obtained from K by replacing the isometric copy of C in K by T. Observe that every vertex $w \in V(K) \setminus V(C)$ is connected to a single vertex in $V(C)$, and has no other neighbors in K. Therefore, the graph T' is a tree with the same distortion as T. It follows K can also be embedded into a distribution $\mathcal{D}'$ over trees with distortion $O(\log g)$.

We are now ready to describe the embedding of G into a random planar graph. We we embed G into a random graph W chosen from $\mathcal{F}$. Recall that W is obtained by 1-sums of a single copy of J with multiple planar graphs. Next, we embed J into a random tree Q chosen from $\mathcal{D}'$, and we replace the isometric copy of J in W by Q. Let R be the resulting graph, illustrated in Figure 2(b). As we already argued, this results in a stochastic $O(g^2)$-embedding. Moreover, the graph R is the 1-sum of a tree with planar graphs. Since the class of planar graphs if closed under 1-sums, it follows that R is planar, concluding the proof. $\square$

3. LOWER BOUNDS

3.1 The dilation of planar planarizing sets

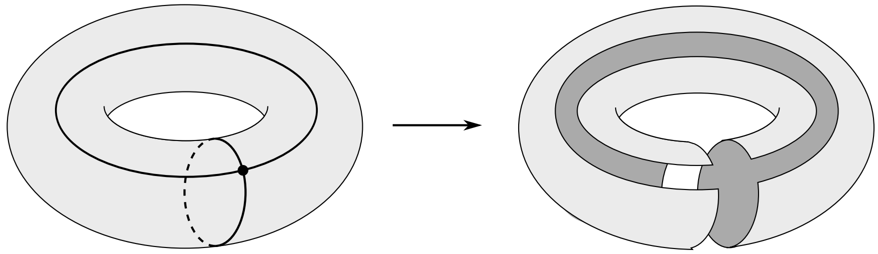
Given the exponential gap on the optimal distortion of a stochastic embedding of a genus-g graph into a distribution over planar graphs ($O(g^2)$ vs. $\Omega(\log g)$), it is natural to ask whether the $O(g^2)$ bound on the distortion of our embedding is tight. It is easy to construct examples of graphs of genus g where a cut graph of minimum total length has dilation $\Omega(g)$. In this case, our embedding clearly has distortion at least $\Omega(g)$, e.g. on the vertices of the planarizing set.

We can in fact show the following lower bound: there are graphs of genus g such that *any* planar planarizing set has dilation $\Omega(g)$. This implies that any algorithm that first computes a planar planarizing set A, and then outputs a stochastic embedding of 1-sums of A with $G \setminus A$ using the Peeling Lemma, has distortion at least $\Omega(g)$.

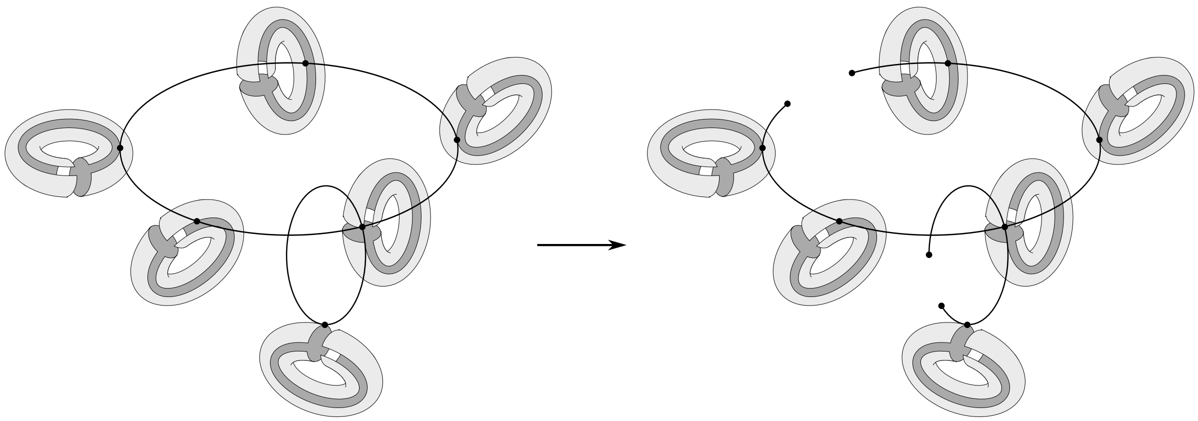
THEOREM 4. *For any $g > 0$, and for any $n > 0$, there exists an n-vertex graph G of genus g, such that for any planar subgraph H of G, and $G \setminus H$ is planar, we have that the dilation of H is at least $\Omega(g)$.*

PROOF. Let S be a surface obtained by K_5 after replacing each vertex by a 3-dimensional sphere of radius 1, and each edge by a cylinder of length n, and radius 1. Let also S' be the surface obtained after attaching $g-1$ handles that are uniformly spread along S (Figure 3). The diameter of each handle is 1. Clearly, the genus of S' is g. Observe that the minimum distance between any two such handles is $\Theta(n/g)$. It is easy to see that we can triangulate S' using triangles of edge-length $\Theta(1)$, such that the set of vertices of the triangulation is a $\Theta(1)$-net of S' of size n.

Let now H be a planar subgraph of G, such that $G \setminus H$ is also planar. Since $G \setminus H$ is planar, it follows that H contains at least one vertex in each handle that we added in S. Let J be the unweighted graph obtained after replacing each edge of a K_5 by a path of length $\frac{g}{100}$. It follows that we can embed J into G with distortion $\Theta(1)$, such that the image of $V(J)$



(a) The graph G with with the planarizing set C, and the graph $G \setminus C$.



(b) The 1-sum of K with copies of $J \setminus K$, and the resulting graph obtained after replacing K with a tree.

Figure 2: Computing the stochastic embedding.

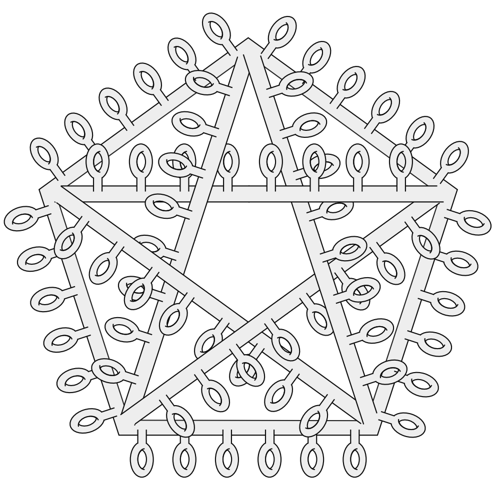


Figure 3: Tori on a surface corresponding to K_5.

is contained in $V(H)$. However, any embedding of J into a planar graph must have distortion $\Omega(g)$. This follows for example by Lemma 1 of [5]. Therefore, the dilation of H is $\Omega(g)$. □

3.2 Lower bounds for randomly planarizing a graph

We now prove the following lower bound.

THEOREM 5. *For every $g \geq 1$, there exists a metric graph $G = (V, E)$ of orientable genus $O(g)$ such that if G admits a stochastic D-embedding into a distribution over planar graph metrics, then $D = \Omega(\log g)$.*

PROOF SKETCH. It is known (and easy to check) that if a metric space (X, d) admits a stochastic D-embedding into a family $\mathcal{Y}$ of metric spaces, then the modulus of decomposability satisfies $\beta_{(X,d)} \leq D \cdot \sup_{(Y,d') \in \mathcal{Y}} \beta_{(Y,d')}$. If $\mathcal{Y}$ is the family of planar graph metrics, then by Theorem 1, the latter quantity is $O(1)$, thus $\beta_{(X,d)} = O(D)$. On the other hand, there are n-point metric spaces (e.g. the shortest-path metric on a constant-degree expander graph) which have $\beta_{(X,d)} = \Omega(\log n)$. Combining this with the fact that every n-point metric space can be represented as the shortest-path metric of a graph with genus at most $O(n^2)$ yields the desired lower bound. □

4. OPEN PROBLEMS

The most immediate problem left open by this work is closing the gap between the $O(g^2)$ upper bound, and the $\Omega(\log g)$ lower bound on the distortion.

Moreover, our embedding yields an algorithm with runnning time $n^{O(g)}$. All the steps of the algorithm can be performed in time $n^{O(1)} \cdot g^{O(1)}$, except for the computation of the minimum-length cut graph, which is NP-complete, and for which the best-known algorithm has running time $n^{O(g)}$ [6]. It remains an interesting open question whether the running time of our algorithm can be improved. We remark that Erickson and Har-Peled [6] also give an algorithm for computing approximate cut graphs, with running time $O(g^2 n \log n)$. These graphs however can have unbounded (in terms of g) dilation, so they don't seem to be applicable in our setting.

Another interesting open problem is whether there exist constant-distortion stochastic embeddings of bounded-genus graphs into planar *subgraphs*.

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