Inapproximability for metric embeddings into R^d

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Metric spaces

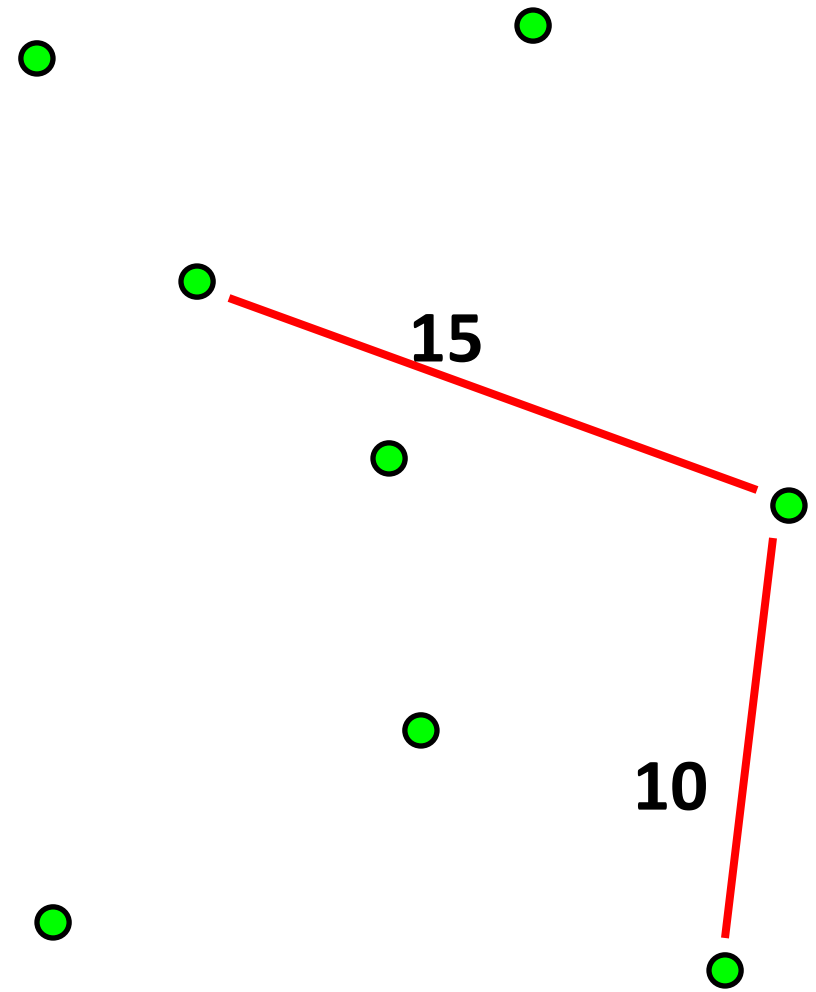
Metric space M=(X,D)

- Positive definiteness

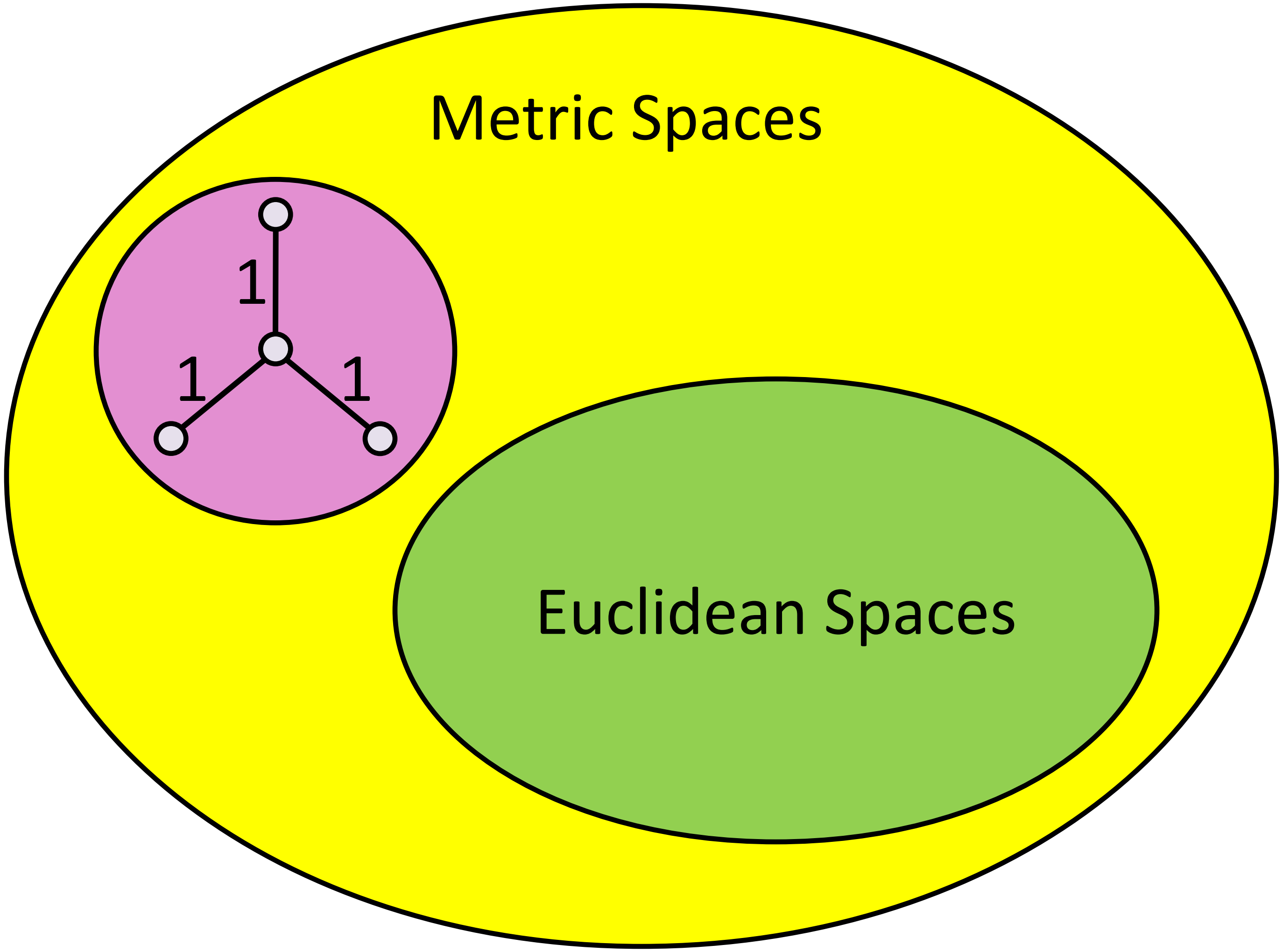
  D(p,q) = 0 iff p = q
- Symmetry

  D(p,q) = D(q,p)
- Triangle inequality

  D(p,q) $\leq$ D(p,r) + D(r,q)

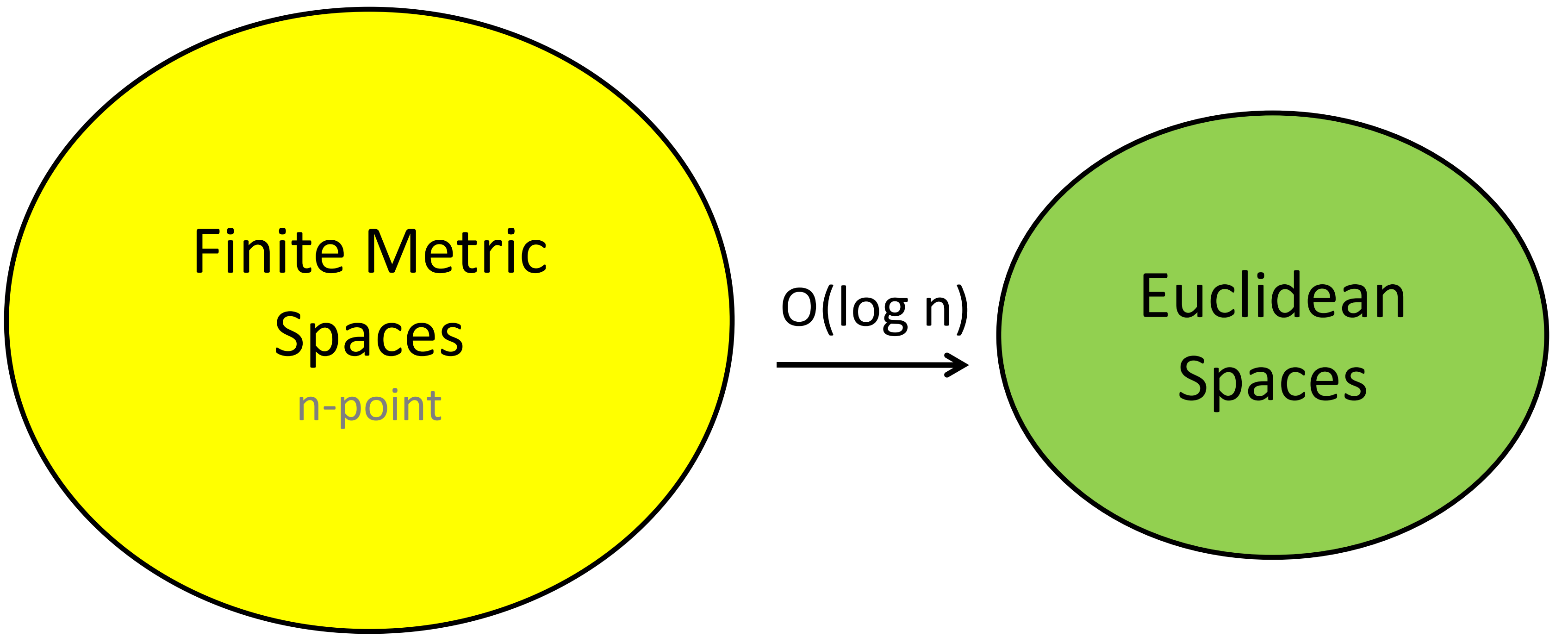


Metric spaces



Metric embeddings

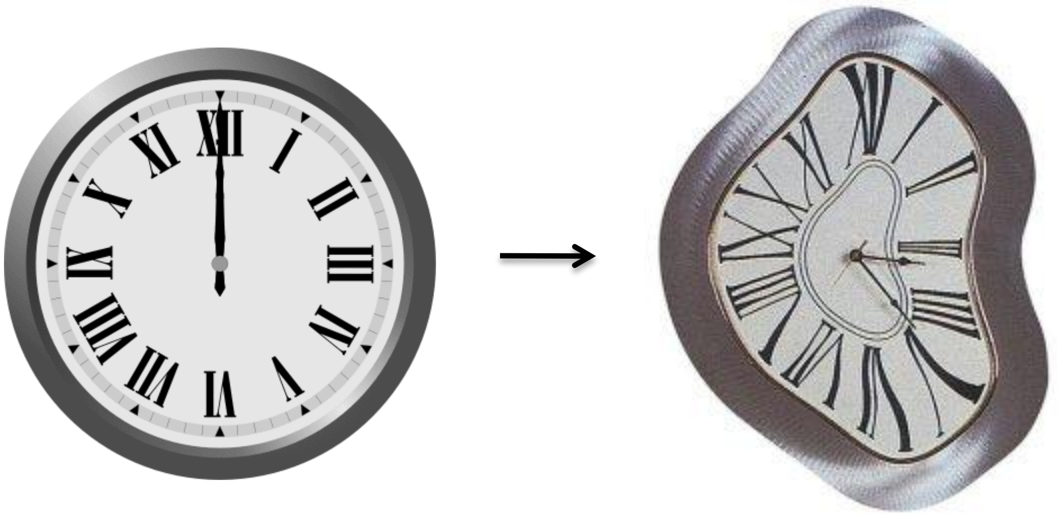
[Bourgain ’85]



Metric embeddings

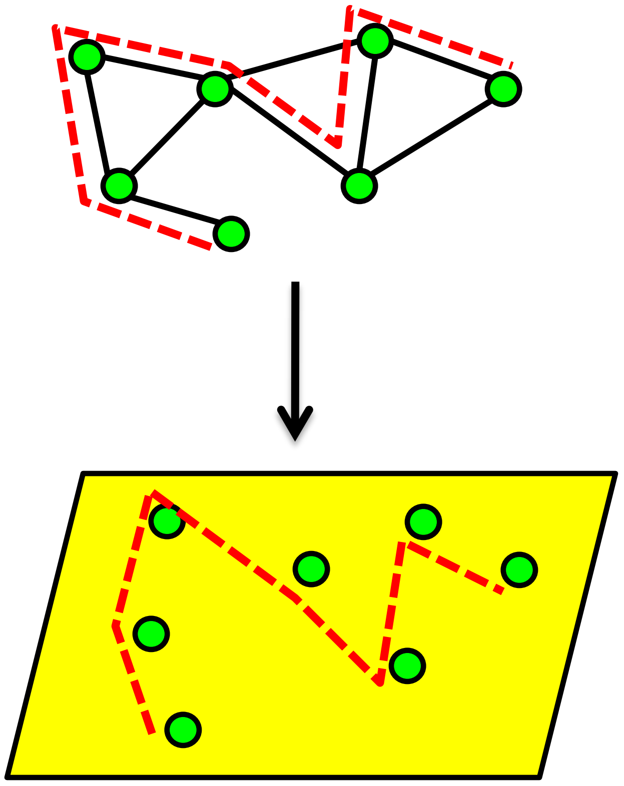
- Given spaces $M=(X,D)$, $M'=(X',D')$
- Mapping $f:X\rightarrow X'$
- Distortion c if:

$$D(x_1,x_2) \le D'(f(x_1),f(x_2)) \le c\cdot D(x_1,x_2)$$



Motivation

- Geometric interpretation
- Succinct data representation
  - Embedding into low-dimensional spaces
- Visualization
  - Embedding into the plane
  - Multi-dimensional scaling
- Optimization
  - Embedding into “easy” spaces

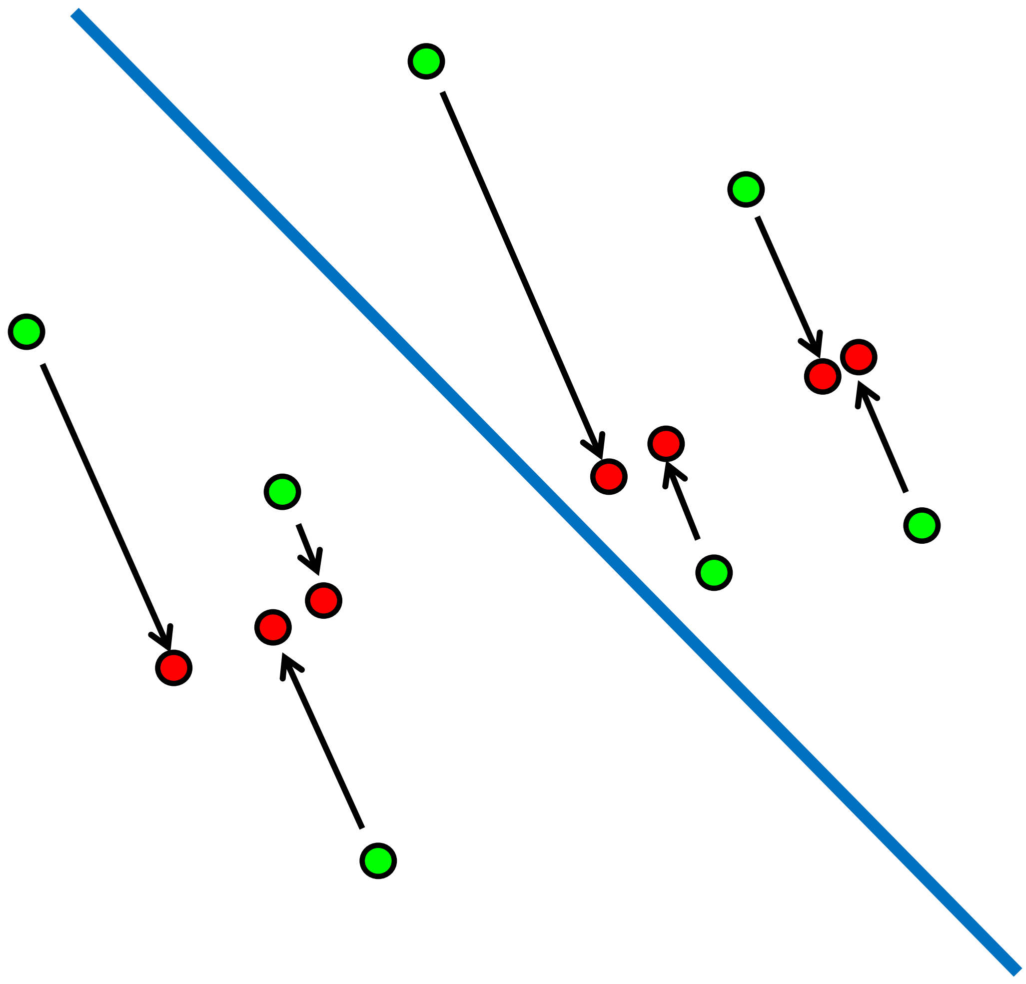


Known results

Host space	**Distortion**	**Citation**
O(log n) –dimensional L_2 (also true for L_p)	O(log n)	[Bourgain '85], [Johnson-Lindenstrauss], [Alon], [Linial, London, Rabinovich '94], [Abraham, Bartal, Neiman '06]
d-dimensional L_2	$\tilde{O}(n^{const/d})$	[Matousek '90] Also: [Gupta '99], [Babilon, Matousek, Maxova, Valtr 2003], [Badoiu, Chuzhoy, Indyk, S '06], [Bateni, Demaine, Hajiaghayi, Moharrami 2007]

Random projection is optimal in the worst case!

Random projection



Absolute vs. Relative embeddings

- Small dimension $\longrightarrow$ high distortion $(n^{\Omega(1/d)})$
  - E.g. embedding a cycle into the line
- What if a particular metric embeds with small distortion?
- Computational problem:

Approximate best possible distortion

Known results on approximation

- Into R^1
  - Unweighted graphs: $n^{1/2}$-approx, 1.01-hard [BDGRRRS '05]
  - Trees: n^{1-a}-approx, n^b-hard [Badoiu, Chuzhoy, Indyk, S '05]
  - General metrics: $(OPT \cdot \log n)^{O(\sqrt{\log \Delta})}$ [Badoiu,Indyk,S'07]
- Into R^d
  - Ultrametrics: $\log^6 \Delta$-approx, NP-hard [Badoiu, Chuzhoy, Indyk, S '06], [Onak, S '08]
  - General metrics: $\tilde{O}(n^{2/d})$ worst case [Matousek '90] $\Omega(n^{1/22d})$-hard [Matousek, S '08]

Random projection is a near-optimal approximation algorithm for general metrics (unless P=NP)!

Reduction outline

Embedding into R^1 $\longrightarrow$ Embedding into R^d

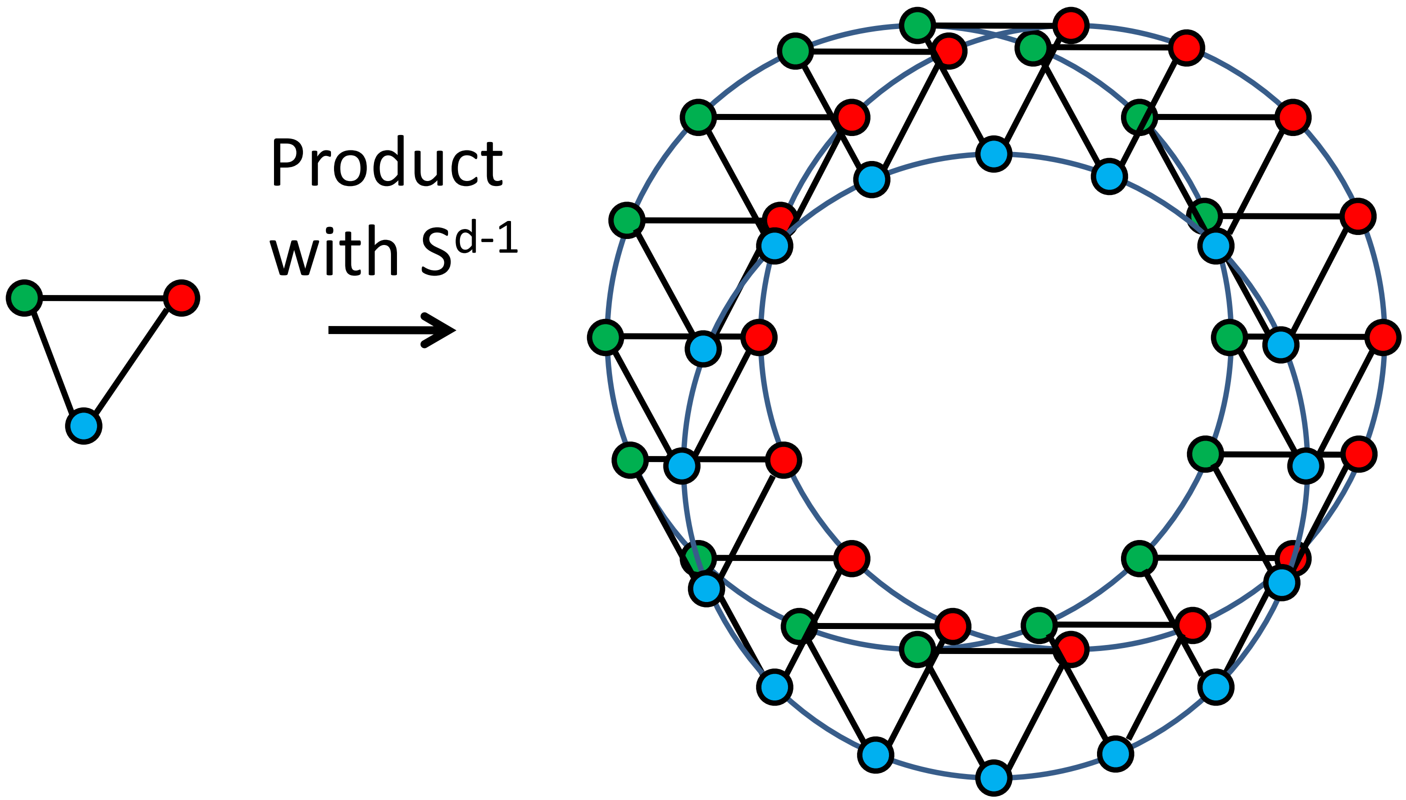
Theorem [Badoiu, Chuzhoy, Indyk, S ‘05]
Embedding into R^1 is NP-hard to approximate within $n^{1/12}$

$\downarrow$

Theorem
For $d \geq 2$,
embedding into R^d is NP-hard to approximate within $n^{1/22d}$

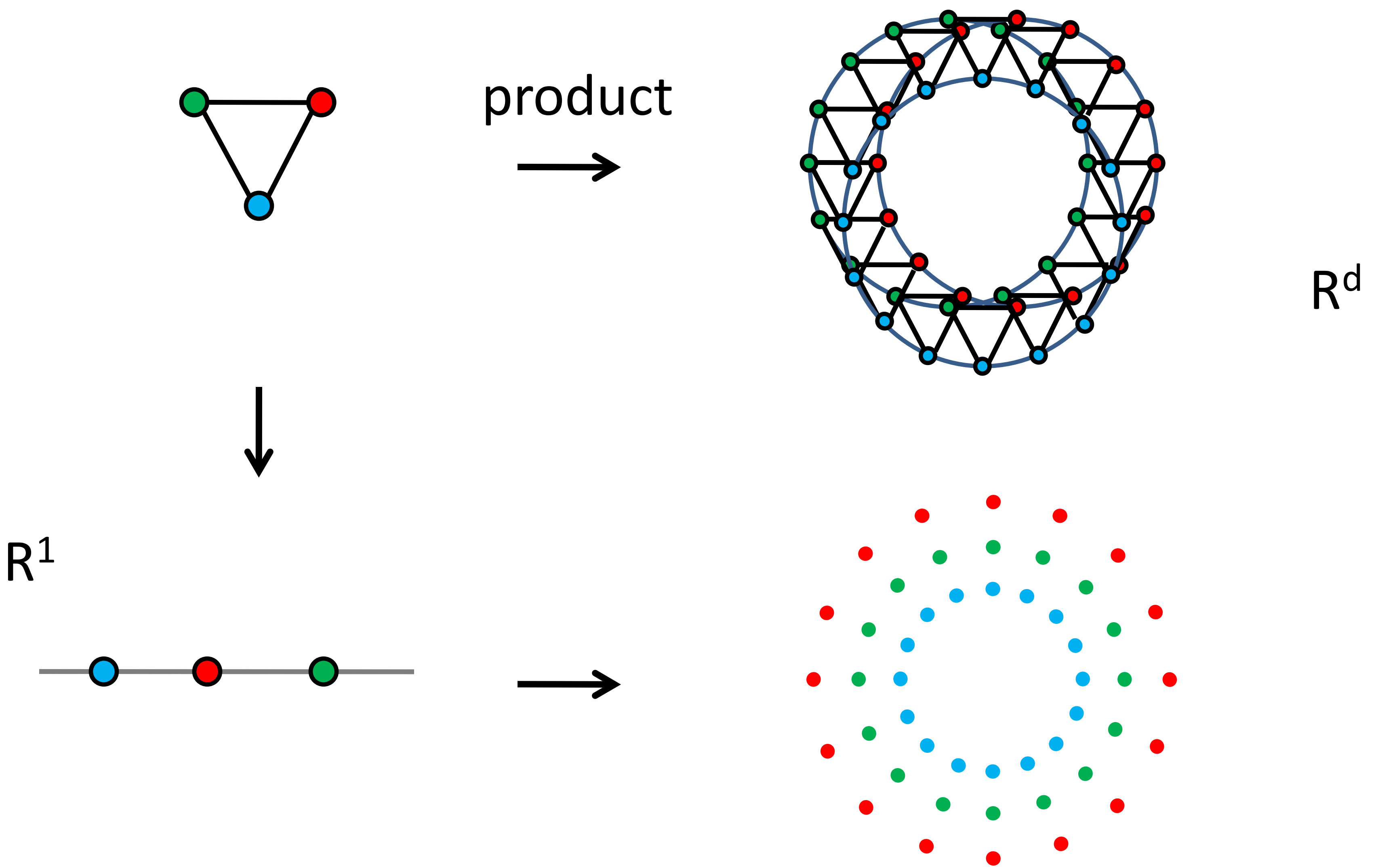
Reduction outline

Reduction from embedding into R^1

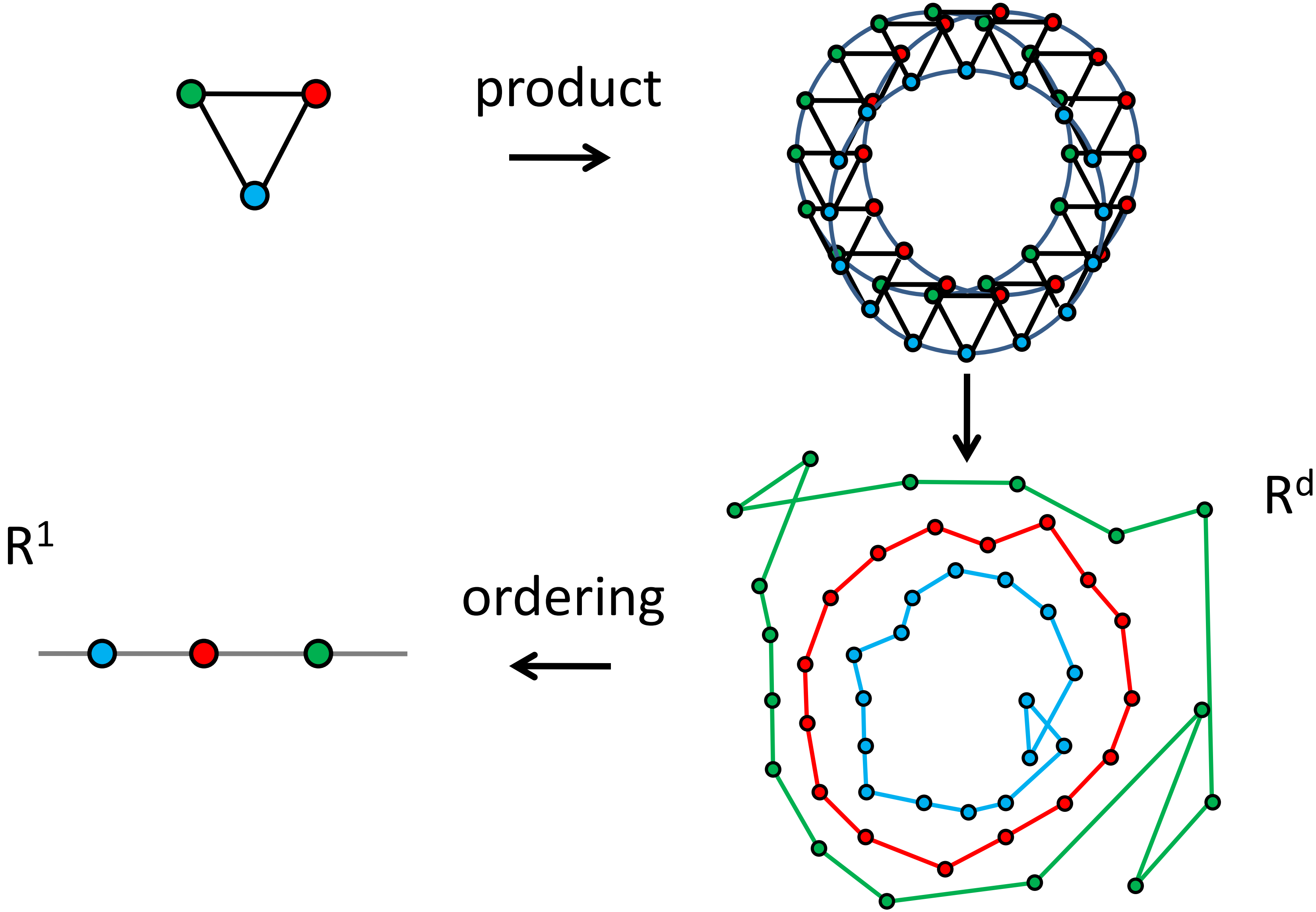


$$M \xrightarrow{c} R^1 \text{ iff } M' \xrightarrow{O(c)} R^d$$

Reduction outline (easy direction)



Reduction outline (hard direction)



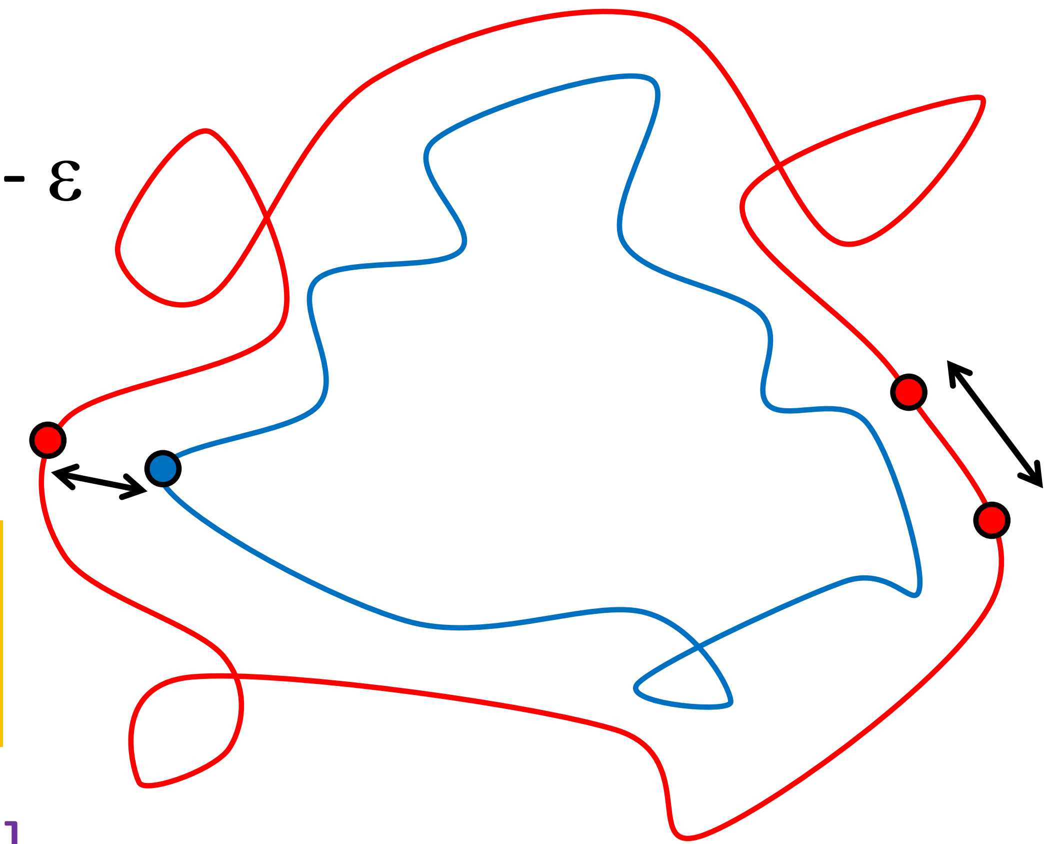
Nesting lemma

$f_1, f_2 : S^{d-1} \longrightarrow R^d$ continuous

- Non-intersecting
- $|f_i(x)-f_i(y)| > |x-y| - \varepsilon$
- $|f_1(x)-f_2(x)| < \varepsilon$

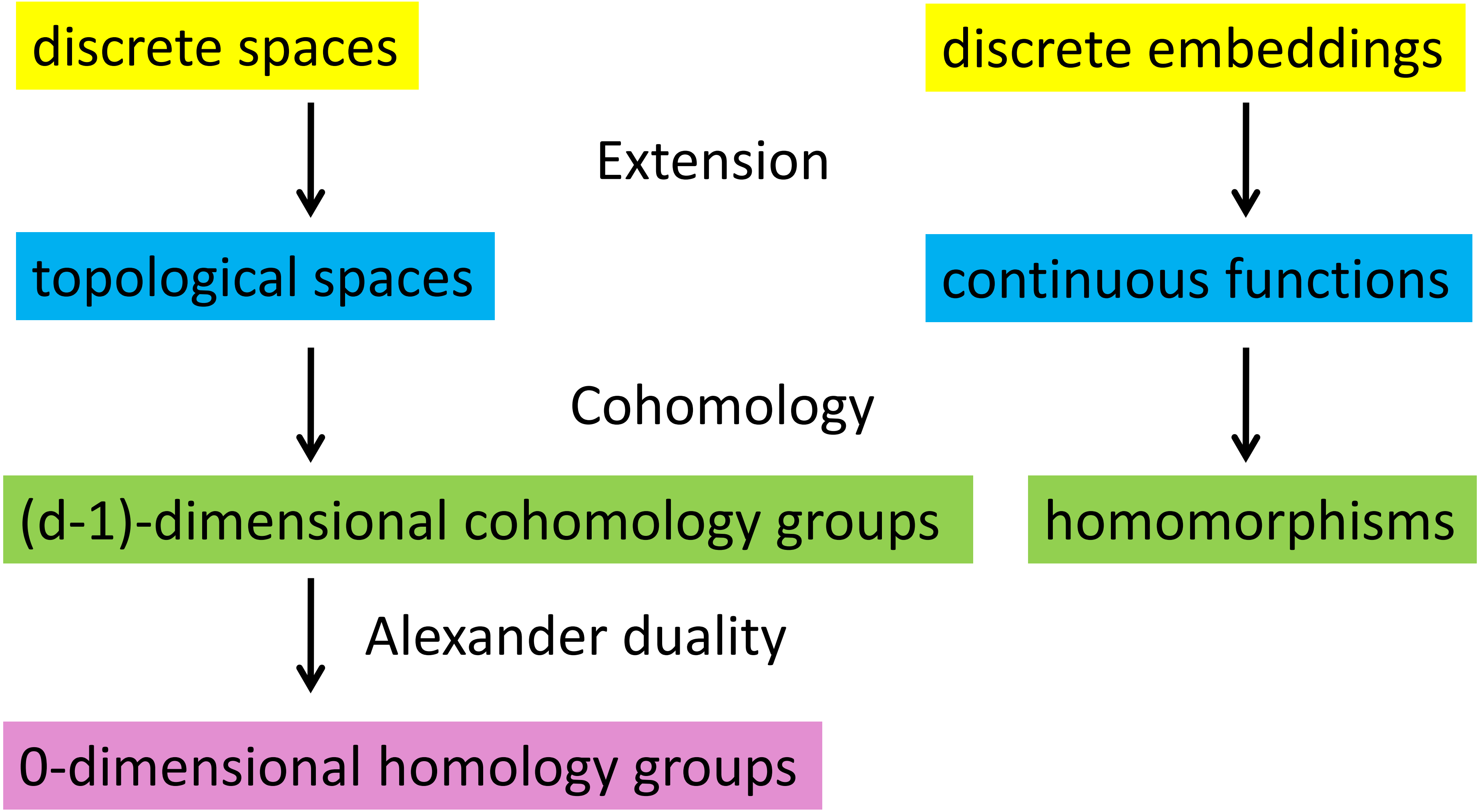
↓

One sphere is "inside" the other!



Ideas from [Vaisala '08]

Proof of nesting lemma: Techniques



What if OPT=O(1)?

- It is NP-hard to distinguish between metrics that embed into R^d with distortion

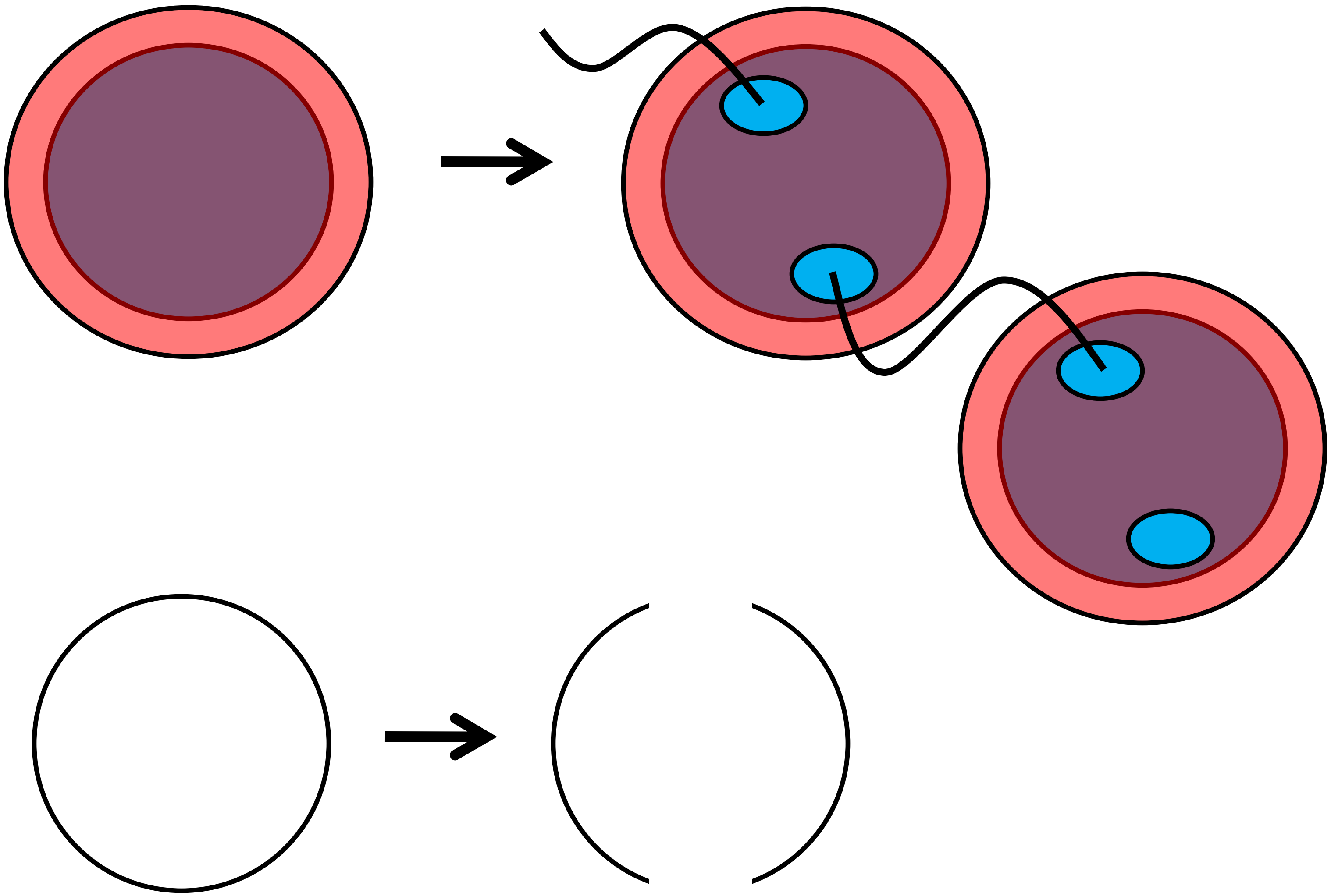
$$n^{a/d} \text{ vs } n^{b/d} \quad (a<b)$$

- Can we distinguish between

$$O(1) \quad \text{vs} \quad n^{b/d}\ ?$$

NO! (for $d \geq 3$)

Improved reduction for d $\geq$ 3



Further directions

- Intriguing open problem:

  Embedding into R^d, $d \leq 2$.

  Is there an algorithm achieving distortion $OPT^{O(1)}$?

- Minimize the dimension.