Optimal stochastic planarization

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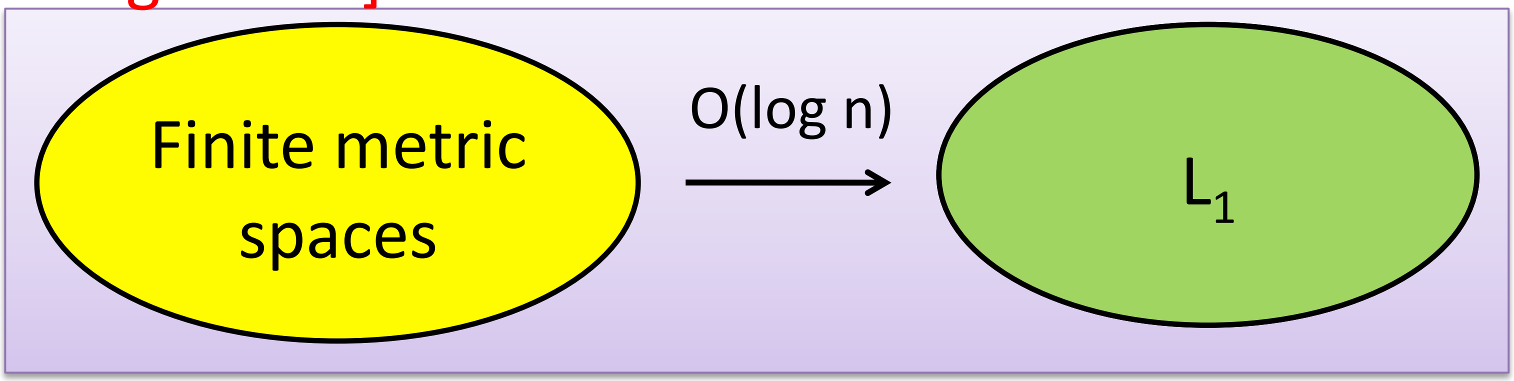
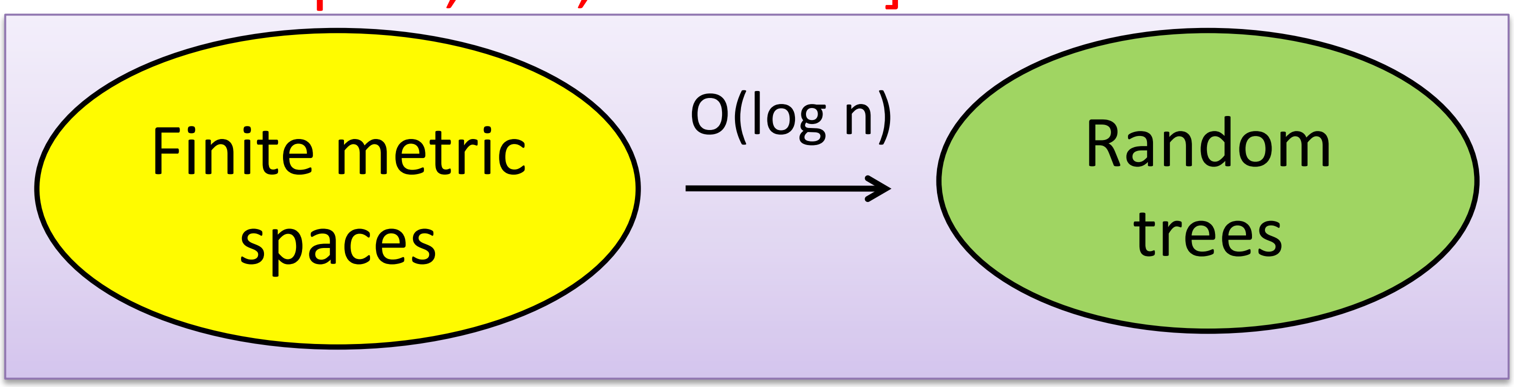
Metric embeddings

[Bourgain '85]



⇑

[Alon,Karp,Peleg,West'91], [Bartal'96], [Bartal'98], [Fakcharoenphol,Rao,Talwar'03]

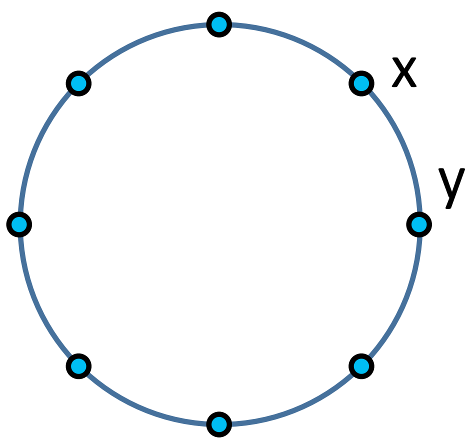


Topological simplification

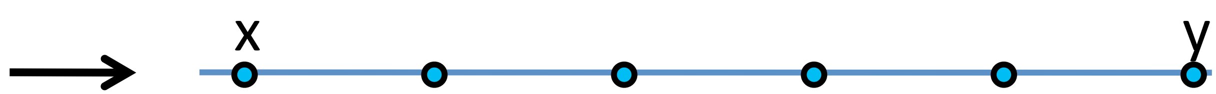
- Topological simplification of a metric space M=(X,D)
- Low distortion embeddings
  - Mapping $f : X \to Y$
  - Preserve distances up to small distortion
- Relaxation: Stochastic embeddings
  - Random mapping $f : X \to Y$
  - Preserve distances in expectation

Stochastic embeddings: example

- Deterministic embedding of the cycle into R^1

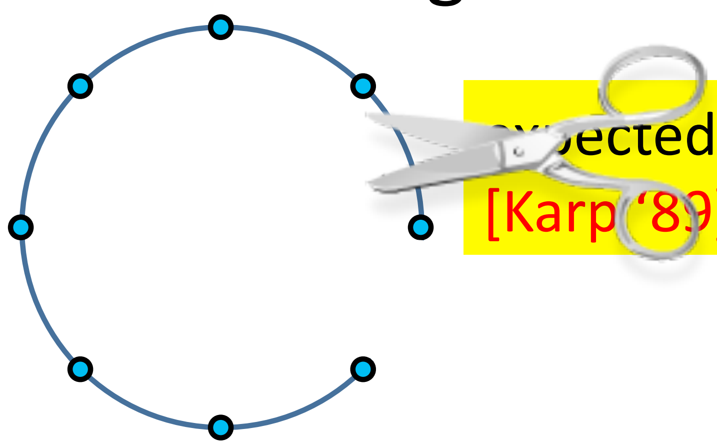


distortion = $\Omega(n)$



- Randomization: Cut an edge at random!

•Pr[edge is cut] = 1/n
•If {x,y} is cut, then D'(x,y)=n-1



expected distortion = O(1)
[Karp '89]

Stochastic embeddings

- Finite metric space $M=(X,D)$
- Distribution $\Phi=\{(M_1,f_1),\ldots,(M_k,f_k)\}$
  - $M_i=(X_i,D_i)$
  - $f_i : X \rightarrow X_i$

  such that $\forall\ u,v \in X$,
  - $\forall\ M_i \in F,\ D_i(u,v) \geq D(u,v)$
  - $\mathbf{E}_{N \in F}\,[D_N(f(u), f(v))] \leq \alpha \cdot D(u,v)$

α : distortion

What about simpler graphs?

- n×n grid ⟶ tree: $\Omega(\log n)$ [Alon,Karp,Peleg,West'91]
- planar ⟶ O(1)-treewidth: $\Omega(\log n)$ [Carroll,Goel'04]
- genus-g ⟶ planar:
  - $2^{O(g)}$ [Indyk, S '07]
  - $g^{O(1)}$ [Borradaile, Lee, S '09]
  - $O(\log g)$ [S '10]
  - $\Omega(\log g)$ [Borradaile, Lee, S '09]

Implications: Approximations algorithms

Let A be a minimization problem, s.t. the objective depends linearly on the distances of the input metric.

(e.g. Distance Oracles, MST, TSP, k-Median, Clustering, Metric Labeling, etc.)

Theorem [S '10]

If there exists an **α**-approx. for A on planar graphs, then there exists an **O(α log g)**-approx. on genus-**g** graphs.

Implications: Sparsest-Cut

gap(G) = $\max_{cap,dem}$ **sparsest-cut** / **max-concurrent-flow**

$\mathbf{c_1}$(G) = inf{c : G embeds into L_1 with distortion c}

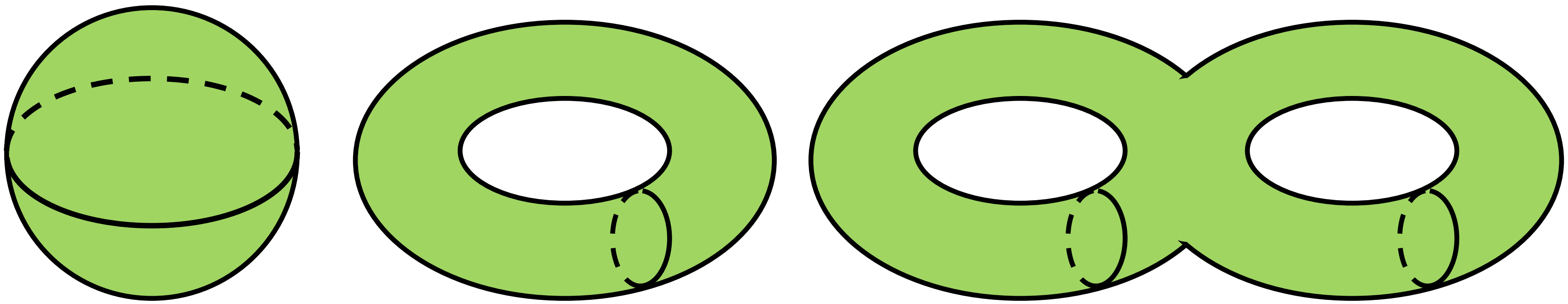
Theorem [Linial,London,Rabinovich'95] [Aumann,Rabani'98]

For every graph G, **gap**(G)=$\mathbf{c_1}$(G)

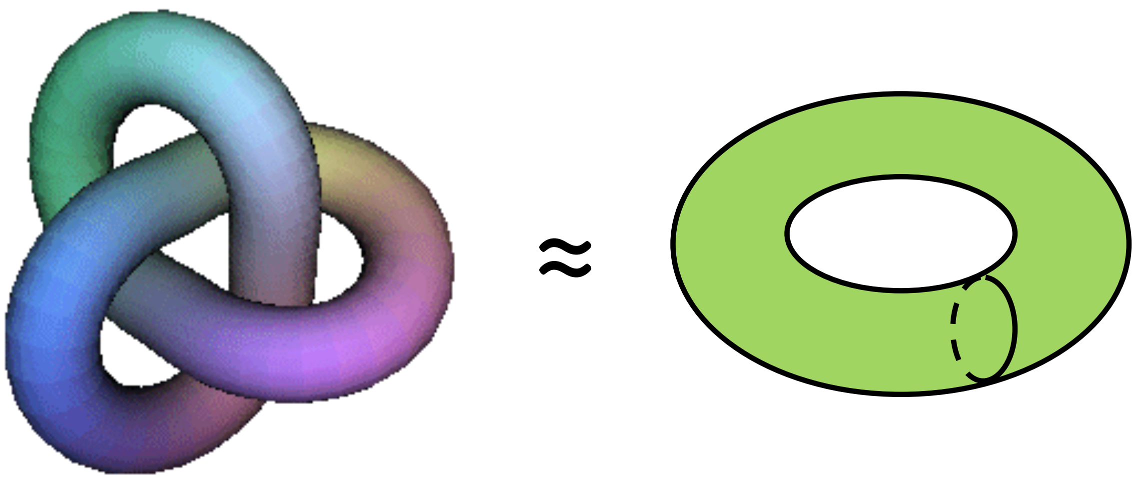
Corollary [Lee, S '09], [S '10]

gap(genus-g) = O(log g) · **gap**(planar)

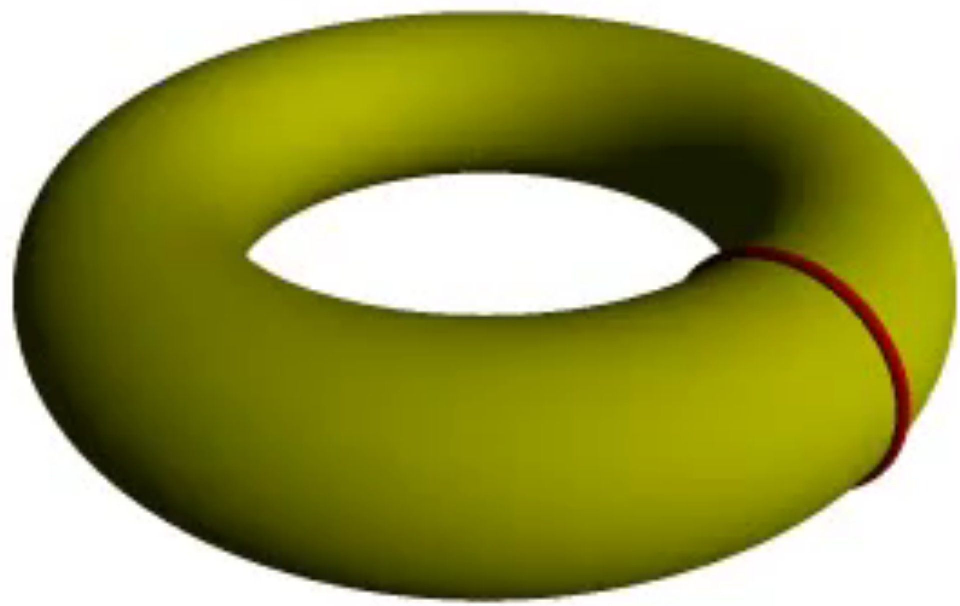
Orientable surfaces



genus 0 genus 1 genus 2



Random cuts



Random cuts

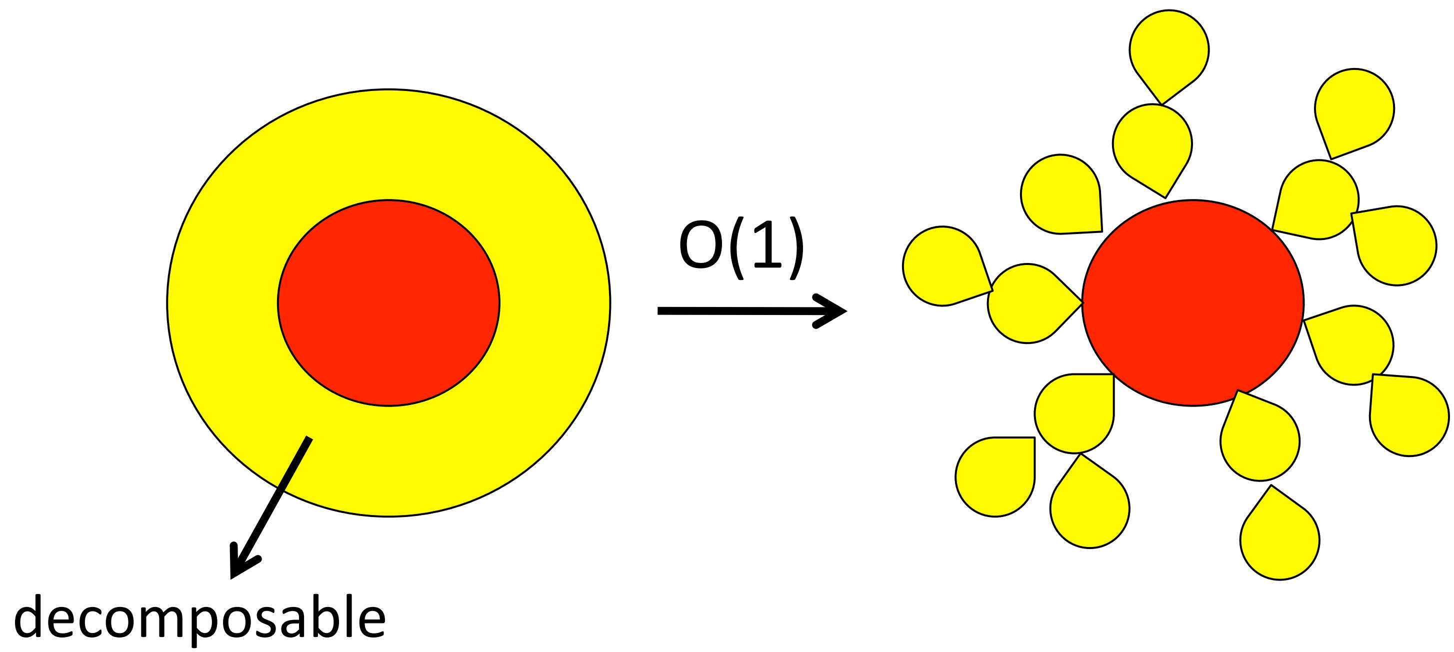
•Pr[edge is cut] = 1/L
•If {x,y} is cut, then D'(x,y)=O(L)

→

expected distortion = O(1)

- Repeating g times gives a planar graph [Indyk,S'07]
- Distortion $2^{O(g)}$

The peeling lemma

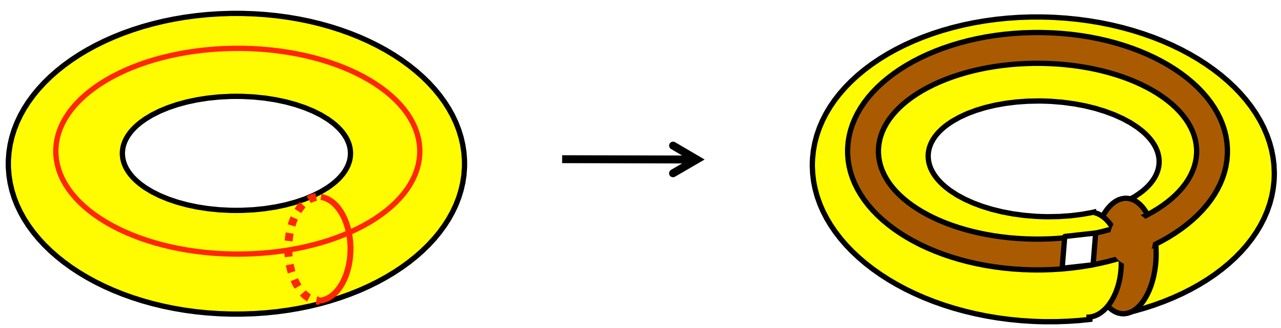


Peeling Lemma [Lee, S ’09]

A ∪ B stochastically O(1)-embeds into 1-sums of A with B

Homotopy generators

- Greedy system of loops [Erickson,Whittlesey'05]
  - Set H of cycles s.t. G\H is planar



Fact: H consists of O(g) shortest paths with a common end-point.

The pathwidth barrier

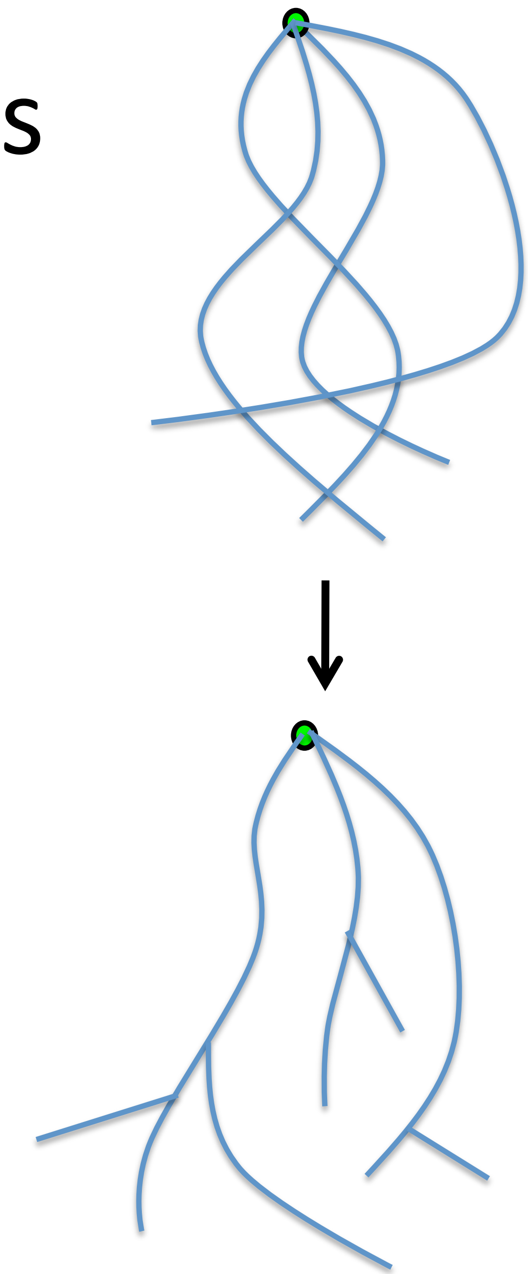
- **Lemma:** [Lee, S ’10]

  The cut graph **H** embeds into a pathwidth-**O(g)** graph,with distortion **O(1)**.

Unfortunately, best-known embedding of pathwidth-**k** graphs into trees has distortion $\mathbf{2^{\Omega(k)}}$. [Lee, S ’09]

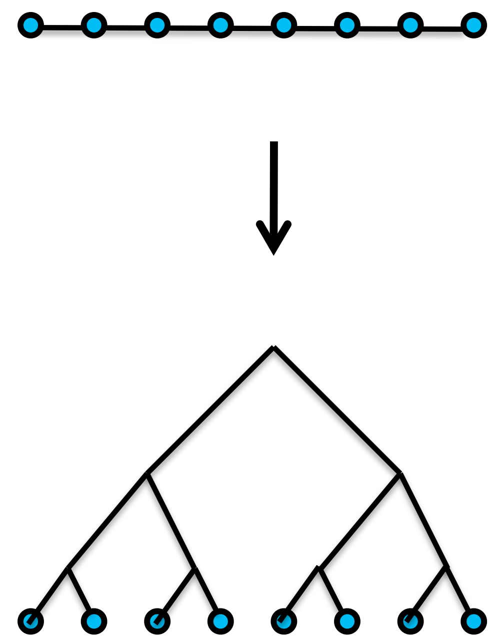
Untangling paths

Theorem: [S ‘10] Let **X** be the union of **g** shortest paths in a graph G, with a common end-point. Then, **(X,d)** embeds into a random tree with distortion **O(log g)**.



The ultrametric barrier

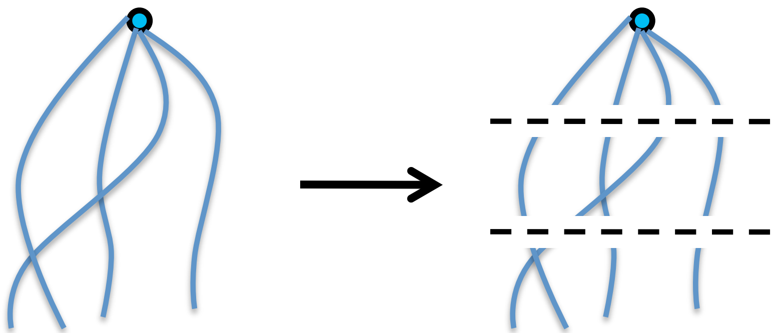
- Essentially all known tree embeddings:
  - Compute a partition for every scale $1,2,4,\ldots,2^i,\ldots$
  - Merge partitions into a tree.
  - The resulting tree is an **ultrametric**.



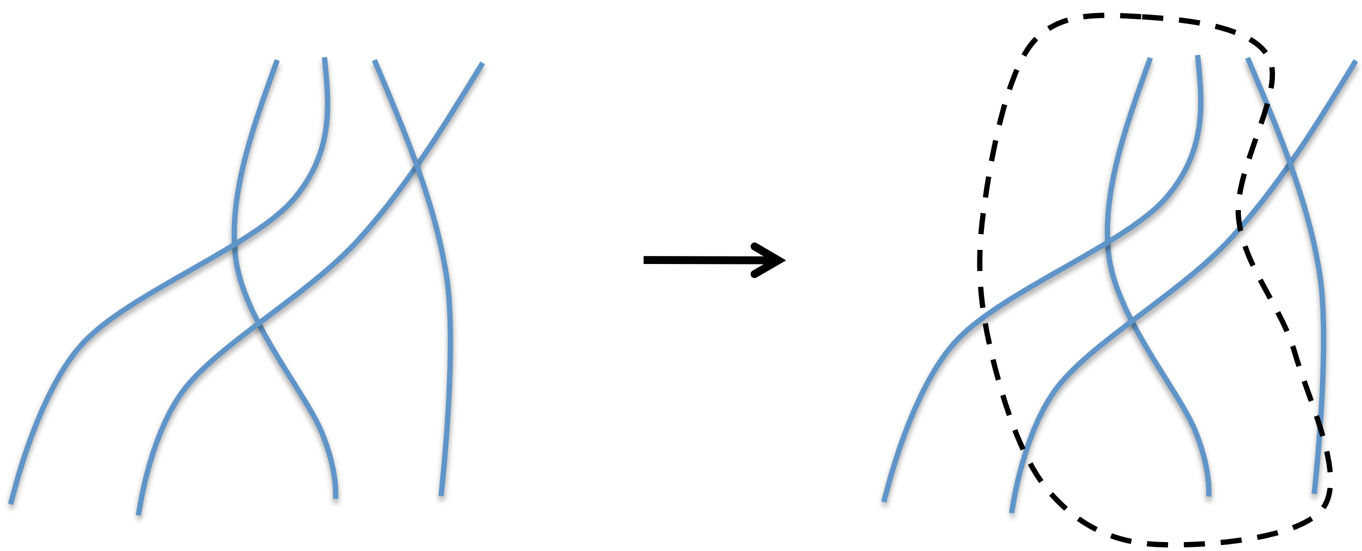
- Any embedding of the **n**-path into a random ultrametric has distortion **Ω(log n)**.

Key idea: Alternating partitions

- Combine two partitions at every scale:
  - **Vertical** partition, similar to [Klein,Plotkin,Rao'93].



  - **Horizontal** partition, similar to [Calinescu,Karloff,Rabani'01].



Open questions

- Genus-**g** into spanning planar subgraphs.
- Pathwidth-**k** into trees with distortion **O(logk)**?
- Optimal embeddings for graphs that exclude a minor **H**, in terms of **|H|**.
  - Only **Ω(log |H|)** lower bounds are known.
  - Almost all upper bounds are super-exponential in **|H|**.